ANHARMONIC OSCILLATORS IN THE COMPLEX PLANE, $\mathcal{PT}$-SYMMETRY, AND REAL EIGENVALUES.

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Abstract. For integers $m \geq 3$ and $1 \leq \ell \leq m-1$, we study the eigenvalue problems $-u''(z) + [(-1)^\ell (iz)^m - P(iz)]u(z) = \lambda u(z)$ with the boundary conditions that $u(z)$ decays to zero as z tends to infinity along the rays $\arg z = -\frac{\pi}{2} \pm \frac{(\ell+1)\pi}{m+2}$ in the complex plane, where P is a polynomial of degree at most $m-1$. We provide asymptotic expansions of the eigenvalues λ_n. Then we show that if the eigenvalue problem is $\mathcal{PT}$-symmetric, then the eigenvalues are all real and positive with at most finitely many exceptions. Moreover, we show that when $\gcd(m, \ell) = 1$, the eigenvalue problem has infinitely many real eigenvalues if and only if its translation or itself is $\mathcal{PT}$-symmetric. Also, we will prove some other interesting direct and inverse spectral results.

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1. Introduction

In this paper, we study Schrödinger eigenvalue problems with real and complex polynomial potentials in the complex plane under various decaying boundary conditions. We provide explicit asymptotic formulas relating the index n to a series of fractional powers of the eigenvalue λ_n (see Theorem 1.1). Also, we recover the polynomial potentials from asymptotic formula of the eigenvalues (see Theorem 1.4 and Corollary 2.3) as well as applications to the so-called $\mathcal{PT}$-symmetric Hamiltonians (see Theorems 1.2 and 1.3).

For integers $m \geq 3$ and $1 \leq \ell \leq m-1$, we consider the Schrödinger eigenvalue problem

$$(H_\ell u)(z) := \left[-\frac{d^2}{dz^2} + (-1)^\ell (iz)^m - P(iz)\right] u(z) = \lambda u(z), \quad \text{for some } \lambda \in \mathbb{C}, \tag{1.1}$$

with the boundary condition that

$$u(z) \to 0 \text{ as } z \to \infty \text{ along the two rays} \quad \arg z = -\frac{\pi}{2} \pm \frac{(\ell+1)\pi}{m+2}, \tag{1.2}$$

where P is a polynomial of degree at most $m-1$ of the form

$$P(z) = a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_{m-1} z + a_m, \quad a_j \in \mathbb{C} \text{ for } 1 \leq j \leq m.$$

If a nonconstant function u satisfies (1.1) with some $\lambda \in \mathbb{C}$ and the boundary condition (1.2), then we call λ an *eigenvalue* of H_ℓ and u an *eigenfunction of H_ℓ associated with the eigenvalue* λ. Sibuya [27] showed that the eigenvalues of H_ℓ are the zeros of an entire function of order $\rho := \frac{1}{2} + \frac{1}{m} \in (0, 1)$ and hence, by the Hadamard factorization theorem (see, e.g., [7, p. 291]), there are infinitely many eigenvalues. We call the entire function the Stokes multiplier

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(or the spectral determinant), and the algebraic multiplicity of an eigenvalue λ is the order of the zero λ of the Stokes multiplier. Also, the geometric multiplicity of an eigenvalue λ is the number of linearly independent eigenfunctions associated with the eigenvalue λ, that is 1 for every eigenvalue λ [16, §7.4].

We number the eigenvalues $\{\lambda_n\}_{n\geq N_0}$ in the order of nondecreasing magnitudes, counting their algebraic multiplicities. We will show that the magnitude of large eigenvalues is strictly increasing (see, Lemma 2.2) and hence, there is a unique way of ordering large eigenvalues, but this is not guaranteed for small eigenvalues. However, how we order these small eigenvalues will not affect results in this paper. Throughout this paper, we will use λ_n to denote the eigenvalues of H_ℓ without explicitly indicating their dependence on the potential and the boundary condition. Also, we let

$$a := (a_1, a_2, \dots, a_m) \in \mathbb{C}^m$$

be the coefficient vector of $P(z)$.

The anharmonic oscillators H_ℓ with the various boundary conditions (1.2) are considered in [4, 24]. When m is even and $\ell = \frac{m}{2}$, $H_{\frac{m}{2}}$ is a Schrödinger operator in $L^2(\mathbb{R})$ (see, e.g., [1, 2, 5, 11, 12, 15, 18, 19]). This is self-adjoint if the potential $V(z) = (-1)^\ell (iz)^m - P(iz)$ is real on the real line, and non-self-adjoint if the potential is non-real.

Some particular classes of H_1 have been studied extensively in recent years in the context of theory of $\mathcal{PT}$-symmetry [3, 6, 10, 14, 20, 26, 28, 29]. The H_ℓ is $\mathcal{PT}$-symmetric if the potential V satisfies $\overline{V(-\overline{z})} = V(z)$, $z \in \mathbb{C}$, that is equivalent to $a \in \mathbb{R}^m$. In this paper, we will generalize results in [26] (where H_1 is studied) to $1 \leq \ell \leq m-1$ and introduce some new results. These results are consequences of the following asymptotic expansion of the eigenvalues.

Theorem 1.1. *For each integer $m \geq 3$ and $1 \leq \ell \leq m-1$, there exists an integer $N_0 = N_0(m,\ell)$ such that the eigenvalues $\{\lambda_n\}_{n\geq N_0}$ of H_ℓ satisfy*

$$n + \frac{1}{2} = \sum_{j=0}^{m+1} c_{\ell,j}(a) \lambda_n^{\frac{1}{2}+\frac{1-j}{m}} + \eta_\ell(a) + O\left(\lambda_n^{-\rho}\right) \quad \text{as } n \to \infty, \tag{1.3}$$

where $c_{\ell,j}(a)$ and $\eta_\ell(a)$ are defined in (2.4) *and* (2.5), *respectively.*

Also, we obtain the partial reality of the eigenvalues for $\mathcal{PT}$-symmetric H_ℓ.

Theorem 1.2. *If H_ℓ is $\mathcal{PT}$-symmetric, then eigenvalues are all real with at most finitely many exceptions.*

Proof. If $u(z)$ is an eigenfunction associated with the eigenvalue λ of a $\mathcal{PT}$-symmetric H_ℓ, then $\overline{u(-\overline{z})}$ is also an eigenfunction associated with $\overline{\lambda}$. In Corollary 2.2, we will show that $|\lambda_n| < |\lambda_{n+1}|$ for all large n and $\arg(\lambda_n) \to 0$. Thus, λ_n are real and positive for all large n since $|\lambda| = |\overline{\lambda}|$. ∎

Many $\mathcal{PT}$-symmetric operators have real eigenvalues only [10, 22, 23]. However, there are some $\mathcal{PT}$-symmetric H_ℓ that produce a finite number of non-real eigenvalues [2, 8, 9, 14]. When H_ℓ is self-adjoint, the spectrum is real. Conversely, when the spectrum is real, what can we conclude about H_ℓ? The next theorem provides a necessary and sufficient condition for H_ℓ to have infinitely many real eigenvalues.

Theorem 1.3. *Suppose that* $\gcd(m,\ell)=1$*. Then* H_ℓ *with the potential* $V(z)=-(iz)^m+P(iz)$ *has infinitely many real eigenvalues if and only if* H_ℓ *with the potential* $V(z-z_0)$ *for some* $z_0\in\mathbb{C}$ *is* $\mathcal{PT}$*-symmetric.*

The next theorem reveals an interesting feature of the eigenvalues as a sequence:

Theorem 1.4. *Suppose that* $\gcd(m,\ell)=1$*. Let* $\{\lambda_n\}_{n\geq N_0}$ *and* $\{\widetilde{\lambda}_n\}_{n\geq N_0}$ *be the eigenvalues of* H_ℓ *with the potentials* V *and* $\widetilde{V}$*, respectively. Suppose that* $\lambda_n-\widetilde{\lambda}_n=o(1)$ *as* $n\to\infty$*. Then* $\widetilde{V}(z)=V(z-z_0)$ *for some* $z_0\in\mathbb{C}$ *and* $\lambda_n=\widetilde{\lambda}_n$ *for all* $n\geq N_0$ *after, if needed, small eigenvalues are reordered.*

The asymptotic expansions of the eigenvalues of H_ℓ with $\ell=\lfloor\frac{m}{2}\rfloor$ have been studied in, for example, [13, 15, 18]. Maslov [18] computed the first three terms of asymptotic expansions of $\lambda_n^{\frac{3}{4}}$, where λ_n are the eigenvalues of $-\frac{d^2}{dx^2}u+x^4u=\lambda u,\ u\in L^2(\mathbb{R})$. Helffer and Robert [15] considered

$$-\frac{d^{2k}}{dx^{2k}}u+(x^{2m}+p(x))u=\lambda u,\quad u\in L^2(\mathbb{R}),$$

where k, m are positive integers and where $p(\cdot)$ is a *real* polynomial of degree at most $2m-1$. They obtained the existence of asymptotic expansions of the eigenvalues to all orders, and suggested an explicit way of computing the coefficients of the asymptotic expansion. In particular, for the case when the potential is εx^4+x^2, $\varepsilon>0$, Helffer and Robert [15] computed the first nine terms of the asymptotic expansion of $\lambda_n^{\frac{3}{4}}$. Also, Fedoryuk [13, §3.3] considered (1.1) with complex polynomial potentials and with (1.2) for $\ell=\lfloor\frac{m}{2}\rfloor$ and computed the first term in the asymptotic expansion. Also, Sibuya [27] computed the first term in the asymptotic expansion for $\ell=1$.

This paper is organized as follows. In Section 2, we define $c_{\ell,j}(a)$ and some other notations. Also, we invert (1.3), expressing λ_n as a series of fractional powers of the index n and prove Theorems 1.3 and 1.4, and other interesting direct and inverse spectral results. In Section 3, we introduce some properties of the solutions of the differential equation in (1.1), due to Hille [16] and Sibuya [27]. We study the asymptotic of the Stokes multiplier associated with H_1 in Section 4 and treat the general case H_ℓ in Section 5. In Section 6, we relate the eigenvalues of H_ℓ with the zeros of the Stokes multiplier. We prove Theorem 1.1 for $1\leq\ell<\frac{m}{2}$ in Section 7 and for $\ell\geq\frac{m}{2}$ in Section 8.

2. Notations and corollaries

In this section, we will define $c_{\ell,j}(a)$ and some other notations and introduce some corollaries of Theorem 1.1.

We define, for nonnegative integers k, j,

$$b_{j,k}(a) \text{ is the coefficient of } \frac{1}{z^j} \text{ in } \binom{\frac{1}{2}}{k}\left(\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots+\frac{a_m}{z^m}\right)^k \quad \text{and} \tag{2.1}$$

$$b_j(a)=\sum_{k=0}^{j} b_{j,k}(a), \quad j\in\mathbb{N}. \tag{2.2}$$

Notice that $b_{0,0}(a)=b_0(a)=1$, $b_{j,0}(a)=0$ if $j\geq 1$, and $b_{j,k}(a)=0$ if $j<k$ or $k<\frac{j}{m}$. We will also use

$$K_{m,j}(a)=\sum_{k=0}^{j} K_{m,j,k}\, b_{j,k}(a) \;\text{ for } j\geq 0, \tag{2.3}$$

where for $k<\frac{j}{m}$, $K_{m,j,k}=0$ and for $\frac{j}{m}\leq k\leq j$,

$$K_{m,j,k}:=\begin{cases} \frac{B\left(\frac{1}{2},1+\frac{1}{m}\right)}{2\cos\left(\frac{\pi}{m}\right)} & \text{if } j=k=0,\\ -\frac{2}{m} & \text{if } j=k=1,\\ \frac{2}{m}\left(\ln 2-\sum_{s=1}^{k-1}\frac{1}{2s-1}\right) & \text{if } j=\frac{m}{2}+1,\ m \text{ even},\\ \frac{1}{m}B\left(k-\frac{j-1}{m},\ \frac{j-1}{m}-\frac{1}{2}\right) & \text{if } j\neq 1 \text{ or } j\neq\frac{m}{2}+1. \end{cases}$$

Now we are ready to define $c_{\ell,j}(a)$ as follows: for $1\leq j\leq m+1$,

$$c_{\ell,j}(a)=-\frac{2}{\pi}\sum_{k=0}^{j}(-1)^{(\ell+1)k}K_{m,j,k}b_{j,k}(a)\sin\left(\frac{(j-1)\ell\pi}{m}\right)\cos\left(\frac{(j-1)\pi}{m}\right), \tag{2.4}$$

and we also define

$$\eta_\ell(a)=\begin{cases} (-1)^{\frac{\ell-1}{2}}\frac{2\nu(a)}{m} & \text{if } \ell \text{ is odd},\\ 0 & \text{if otherwise} \end{cases} \quad \text{and} \quad \mu(a)=\frac{m}{4}-\nu(a), \tag{2.5}$$

where

$$\nu(a)=\begin{cases} b_{\frac{m}{2}+1}(a) & \text{if } m \text{ is even},\\ 0 & \text{if } m \text{ is odd}. \end{cases}$$

2.1. **Further direct spectral results.** One can invert the asymptotic formulas (1.3) to obtain formulas for λ_n in terms of n.

Corollary 2.1. *One can compute $d_{\ell,j}(a)$ explicitly such that*

$$\lambda_n = \sum_{j=0}^{m+1} d_{\ell,j}(a)\left(n+\frac{1}{2}\right)^{\frac{2m}{m+2}\left(1-\frac{j}{m}\right)} + O\left(n^{-\frac{4}{m+2}}\right), \tag{2.6}$$

where $d_{\ell,0}(a) = \left(\pi^{-1} B\left(\frac{1}{2}, 1+\frac{1}{m}\right)\sin\left(\frac{\ell\pi}{m}\right)\right)^{-\frac{2m}{m+2}} > 0.$

Proof. Equations (1.3) is an asymptotic equation and it can be solved for λ_n, resulting in (2.6). For details, see, for example, [24, §5]. □

In the next corollary, we provide an asymptotic formula for the nearest neighbor spacing of the eigenvalues. And the large eigenvalues increase monotonically in magnitude and have their argument approaching zero.

Corollary 2.2. *The space between successive eigenvalues is*

$$\lambda_{n+1} - \lambda_n \underset{n\to+\infty}{=} \frac{2m}{m+2}\, d_{\ell,0}\cdot\left(n+\frac{1}{2}\right)^{\frac{m-2}{m+2}} + o\left(n^{\frac{m-2}{m+2}}\right). \tag{2.7}$$

In particular, $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n| = \infty$ *and* $\lim_{n\to+\infty}\arg(\lambda_n) = 0$. *Hence:*

$$|\lambda_n| < |\lambda_{n+1}| \quad \textit{for all large } n.$$

Proof. These claims are consequences of (2.6) and the generalized binomial expansion. For details, see, for example, [24, §6]. □

Suppose that λ_n are eigenvalues of H_ℓ for some P. Then the degree m of the polynomial potential can be recovered by

$$\frac{2m}{m+2} = \lim_{n\to\infty}\frac{\ln(|\lambda_n|)}{\ln n} < 2.$$

The next corollary shows that one can recover the polynomial potential from the asymptotic formula for the eigenvalues.

Corollary 2.3. *Suppose that* $\gcd(m,\ell) = 1$ *and that* $\{\lambda_n\}_{n\ge N_0}$ *are the eigenvalues of* H_ℓ *with the potential* V. *If*

$$\sum_{j=0}^{m+1} c_j^* \lambda_n^{\frac{1}{2}+\frac{1-j}{m}} + O\left(\lambda_n^{-\frac{1}{2}-\frac{1}{m}}\right) = \left(n+\frac{1}{2}\right) \tag{2.8}$$

as $n\to\infty$ *for some* $c_j^*\in\mathbb{C}$, *or if*

$$\lambda_n = \sum_{j=0}^{m+1} d_j^*\left(n+\frac{1}{2}\right)^{\frac{2m}{m+2}\left(1-\frac{j}{m}\right)} + O\left(n^{-\frac{4}{m+2}}\right), \tag{2.9}$$

for some $d_j^*\in\mathbb{C}$, *then there exists* V_0 *such that for each polynomial* V *with which* H_ℓ *generates* $\{\lambda_n\}_{n\ge N_0}$, $V(z) = V_0(z-z_0)$ *for some* $z_0\in\mathbb{C}$.

Proof. The coefficients $c_{\ell,j}(a)$ and $d_{\ell,j}(a)$ have the following properties.

(i) The $c_{\ell,j}(a)$ and $d_{\ell,j}(a)$ are all *real* polynomials in terms of the coefficients a of $P(x)$.

(ii) The coefficients $c_{\ell,0}(a)$ and $d_{\ell,0}(a)$ do not depend on a (they are constants).

(iii) For $1 \leq j \leq m$, the polynomials $c_{\ell,j}(a)$ and $d_{\ell,j}(a)$ depend only on $a_1, a_2, \ldots, a_j$. Furthermore, if $\gcd(m,\ell)=1$ and if $j \neq 1, \frac{m}{2}+1$, then $c_{\ell,j}(a)$ and $d_{\ell,j}(a)$ are non-constant linear functions of a_j.

Here we will sketch the proof when (2.8) is assumed. First, notice that $c_1^*=0$. Otherwise, λ_n are not the eigenvalues of H_ℓ, according to (2.4). Choose $a_1=0$. Then since $c_{\ell,2}(a)$ is a non-constant linear function of a_2 and depends on a_1 and a_2, $c_{\ell,2}(a)=c_2^*$ determines a_2. If m is even, $c_{\ell,\frac{m}{2}+1}(a)=0$ and we use $\eta_\ell(a)$ instead of $c_{\ell,\frac{m}{2}+1}(a)$. Suppose that for $2 \leq j \leq m-1$, the $c_2^*, \ldots, c_j^*$ uniquely determine $a_2, a_3, \ldots, a_j$. Then by (iii), $c_{\ell,j+1}(a)=c_{j+1}^*$ determines a_{j+1} uniquely. So by induction, one can determine all a_j for $2 \leq j \leq m$ and V_0 is the potential with these a_j's. Then $V(z)=V_0(z-z_0)$ for some $z_0 \in \mathbb{C}$. Otherwise, the eigenvalues $\{\lambda_n\}$ do not satisfy (2.8).

The case when (2.9) holds can be handled similarly. □

Next, we will prove Theorems 1.3 and 1.4.

Proof of Theorems 1.3. If $V(z-z_0)$ for some $z_0 \in \mathbb{C}$ is $\mathcal{PT}$-symmetric, then by Theorem 1.2, all but finitely many eigenvalues are real and hence, there are infinitely many real eigenvalues.

Suppose that H_ℓ with the potential $V(z)$ has infinitely many real eigenvalues. Then we can always find $z_0 \in \mathbb{C}$ so that $V(z-z_0)$ has no z^{m-1}-term, that is, $a_1=0$. We will show that $V(z-z_0)$ is $\mathcal{PT}$-symmetric. Since there are infinitely many real eigenvalues, $c_{\ell,j}(a)$, $0 \leq j \leq m+1$, and η_ℓ in (1.3) are real. Also, the sine term in (2.4) does not vanish except for $j=1$. And the cosine term does not vanish except for $j=\frac{m}{2}+1$ when m is even in which η_ℓ replaces the role of $c_{\ell,\frac{m}{2}}$.

Like we did for proof of Corollary 2.3, since $a_1=0$, from the above properties of $c_{\ell,j}$ and η_ℓ, by the induction, we can show that a_j for $2 \leq j \leq m$ are all real and hence, $V(z-z_0)$ is $\mathcal{PT}$-symmetric. □

Proof of Theorems 1.4. Suppose that $\lambda_n-\widetilde{\lambda}_n=o(1)$ as $n\to\infty$. Then $d_{\ell,j}(a)=d_{\ell,j}(\widetilde{a})$ for all $1 \leq j \leq m+1$. Then Corollary 2.3 completes proof. □

3. Properties of the solutions

In this section, we introduce work of Hille [16] and Sibuya [27] about properties of the solutions of (1.1).

First, we scale equation (1.1) because many facts that we need later are stated for the scaled equation. Let u be a solution of (1.1) and let $v(z)=u(-iz)$. Then v solves

$$-v''(z)+[(-1)^{\ell+1}z^m+P(z)+\lambda]v(z)=0. \tag{3.1}$$

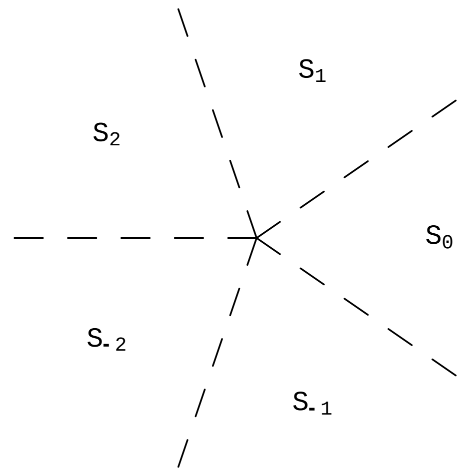

FIGURE 1. The Stokes sectors for $m = 3$. The dashed rays represent $\arg z = \pm\frac{\pi}{5}, \pm\frac{3\pi}{5}, \pi$.

When ℓ is odd, (3.1) becomes

$$-v''(z) + [z^m + P(z) + \lambda]v(z) = 0. \tag{3.2}$$

Later we will handle the case when ℓ is even.

Since we scaled the argument of u, we must rotate the boundary conditions. We state them in a more general context by using the following definition.

Definition. *The Stokes sectors* S_k of the equation (3.2) are

$$S_k = \left\{ z \in \mathbb{C} : \left| \arg(z) - \frac{2k\pi}{m+2} \right| < \frac{\pi}{m+2} \right\} \quad \text{for} \quad k \in \mathbb{Z}.$$

See Figure 1.

It is known from Hille [16, §7.4] that every nonconstant solution of (3.2) either decays to zero or blows up exponentially, in each Stokes sector S_k.

Lemma 3.1 ([16, §7.4])**.**

(i) *For each $k \in \mathbb{Z}$, every solution v of (3.2) is asymptotic to*

$$(const.)z^{-\frac{m}{4}} \exp\left[\pm \int^z [\xi^m + P(\xi) + \lambda]^{\frac{1}{2}}\, d\xi\right] \tag{3.3}$$

as $z \to \infty$ in every closed subsector of S_k.

(ii) *If a nonconstant solution v of* (3.2) *decays in S_k, it must blow up in $S_{k-1} \cup S_{k+1}$. However, when v blows up in S_k, v need not be decaying in S_{k-1} or in S_{k+1}.*

Lemma 3.1 (i) implies that if v decays along one ray in S_k, then it decays along all rays in S_k. Also, if v blows up along one ray in S_k, then it blows up along all rays in S_k. Thus, the boundary conditions (1.2) with $1 \le \ell \le m-1$ represent all decaying boundary conditions.

Still with ℓ odd, the two rays in (1.2) map, by $z \mapsto -iz$, to the rays $\arg(z) = \pm\frac{(\ell+1)\pi}{m+2}$ which are the center rays of the Stokes sectors $S_{\frac{\ell+1}{2}}$ and $S_{-\frac{\ell+1}{2}}$ and the boundary conditions

(1.2) on u become

$$v \text{ decays to zero in the Stokes sector } S_{\frac{\ell+1}{2}} \text{ and } S_{-\frac{\ell+1}{2}}.$$

When ℓ is even, we let $y(z) = v(\omega^{-\frac{1}{2}}z)$ so that (3.1) becomes

$$-y''(z) + [z^m + \omega^{-1}P(\omega^{-\frac{1}{2}}z) + \omega^{-1}\lambda]y(z) = 0, \tag{3.4}$$

where

$$\omega = \exp\left[\frac{2\pi i}{m+2}\right]$$

and hence, $\omega^{-\frac{m}{2}+1} = -1$. For these cases, the boundary conditions (1.2) become

$$y \text{ decays to zero in the Stokes sector } S_{\frac{\ell+2}{2}} \text{ and } S_{-\frac{\ell}{2}}.$$

The following theorem is a special case of Theorems 6.1, 7.2, 19.1 and 20.1 of Sibuya [27] that is the main ingredient of the proofs of the main results in this paper. For this we will use $r_m = -\frac{m}{4}$ if m is odd, and $r_m = -\frac{m}{4} - b_{\frac{m}{2}+1}(a)$ if m is even.

Theorem 3.2. *Equation (3.2), with $a \in \mathbb{C}^m$, admits a solution $f(z, a, \lambda)$ with the following properties.*

(i) *$f(z, a, \lambda)$ is an entire function of z, a and λ.*

(ii) *$f(z, a, \lambda)$ and $f'(z, a, \lambda) = \frac{\partial}{\partial z} f(z, a, \lambda)$ admit the following asymptotic expansions. Let $\varepsilon > 0$. Then*

$$\begin{aligned} f(z, a, \lambda) &= \quad z^{r_m}(1 + O(z^{-1/2})) \exp\left[-F(z, a, \lambda)\right], \\ f'(z, a, \lambda) &= -\, z^{r_m+\frac{m}{2}}(1 + O(z^{-1/2})) \exp\left[-F(z, a, \lambda)\right], \end{aligned}$$

as z tends to infinity in the sector $|\arg z| \leq \frac{3\pi}{m+2} - \varepsilon$, uniformly on each compact set of (a, λ)-values . Here

$$F(z, a, \lambda) = \frac{2}{m+2} z^{\frac{m}{2}+1} + \sum_{1 \leq j < \frac{m}{2}+1} \frac{2}{m+2-2j} b_j(a) z^{\frac{1}{2}(m+2-2j)}.$$

(iii) *Properties* (i) *and* (ii) *uniquely determine the solution $f(z, a, \lambda)$ of (3.2).*

(iv) *For each fixed $a \in \mathbb{C}^m$ and $\delta > 0$, f and f' also admit the asymptotic expansions,*

$$f(0, a, \lambda) = [1 + O\left(\lambda^{-\rho}\right)]\lambda^{-1/4} \exp\left[L(a, \lambda)\right], \tag{3.5}$$

$$f'(0, a, \lambda) = -\,[1 + O\left(\lambda^{-\rho}\right)]\lambda^{1/4} \exp\left[L(a, \lambda)\right], \tag{3.6}$$

as $\lambda \to \infty$ in the sector $|\arg(\lambda)| \leq \pi - \delta$, uniformly on each compact set of $a \in \mathbb{C}^m$, where

$$L(a, \lambda) = \begin{cases} \int_0^{+\infty} \left(\sqrt{t^m + P(t) + \lambda} - t^{\frac{m}{2}} - \sum_{j=1}^{\frac{m+1}{2}} b_j(a) t^{\frac{m}{2}-j}\right) dt & \text{if } m \text{ is odd}, \\ \int_0^{+\infty} \left(\sqrt{t^m + P(t) + \lambda} - t^{\frac{m}{2}} - \sum_{j=1}^{\frac{m}{2}} b_j(a) t^{\frac{m}{2}-j} - \frac{b_{\frac{m}{2}+1}}{t+1}\right) dt & \text{if } m \text{ is even}. \end{cases}$$

(v) *The entire functions $\lambda \mapsto f(0, a, \lambda)$ and $\lambda \mapsto f'(0, a, \lambda)$ have orders $\frac{1}{2} + \frac{1}{m}$.*

Proof. In Sibuya's book [27], see Theorem 6.1 for a proof of (i) and (ii); Theorem 7.2 for a proof of (iii) with the error terms $o(1)$; and Theorem 19.1 for a proof of (iv). Moreover, (v) is a consequence of (iv) along with Theorem 20.1 in [27]. The error terms in (iii) are improved from $o(1)$ to $O(\lambda^{-\rho})$ in [25]. Note that properties (i), (ii) and (iv) are summarized on pages 112–113 of Sibuya [27]. □

Remarks. Throughout this paper, we will deal with numbers like $(\omega^\alpha\lambda)^s$ for some $s \in \mathbb{R}$ and $\alpha \in \mathbb{C}$. As usual, we will use

$$\omega^\alpha = \exp\left[\alpha\frac{2\pi i}{m+2}\right]$$

and if $\arg(\lambda)$ is specified, then

$$\arg\left((\omega^\alpha\lambda)^s\right) = s\left[\arg(\omega^\alpha) + \arg(\lambda)\right] = s\left[\operatorname{Re}(\alpha)\frac{2\pi}{m+2} + \arg(\lambda)\right], \quad s \in \mathbb{R}.$$

Lemma 3.3. *Let $m \geq 3$ and $a \in \mathbb{C}^m$ be fixed. Then*

$$L(a,\lambda) = \sum_{j=0}^{\infty} K_{m,j}(a)\lambda^{\frac{1}{2}+\frac{1-j}{m}} - \frac{\nu(a)}{m}\ln(\lambda) \tag{3.7}$$

as $\lambda \to \infty$ in the sector $|\arg(\lambda)| \leq \pi - \delta$, uniformly on each compact set of $a \in \mathbb{C}^m$.

Proof. See [24] for a proof. □

Sibuya [27] proved the following corollary, directly from Theorem 3.2, that will be used later in Sections 4 and 5.

Corollary 3.4. *Let $a \in \mathbb{C}^m$ be fixed. Then $L(a,\lambda) = K_m\lambda^{\frac{1}{2}+\frac{1}{m}}(1+o(1))$ as λ tends to infinity in the sector $|\arg\lambda| \leq \pi-\delta$, and hence*

$$\operatorname{Re}\left(L(a,\lambda)\right) = K_m\cos\left(\frac{m+2}{2m}\arg(\lambda)\right)|\lambda|^{\frac{1}{2}+\frac{1}{m}}(1+o(1)) \tag{3.8}$$

as $\lambda \to \infty$ in the sector $|\arg(\lambda)| \leq \pi - \delta$.

In particular, $\operatorname{Re}\left(L(a,\lambda)\right) \to +\infty$ *as $\lambda \to \infty$ in any closed subsector of the sector $|\arg(\lambda)| < \frac{m\pi}{m+2}$. In addition,* $\operatorname{Re}\left(L(a,\lambda)\right) \to -\infty$ *as $\lambda \to \infty$ in any closed subsector of the sectors $\frac{m\pi}{m+2} < |\arg(\lambda)| < \pi - \delta$.*

Based on Corollary 3.4, Sibuya [27, Thm. 29.1] also computed the leading term in (1.3) for $\ell = 1$. Also, Sibuya [27] constructed solutions of (3.2) that decays in S_k, $k \in \mathbb{Z}$. Before we introduce this, we let

$$G^\ell(a) := (\omega^{(m+1)\ell}a_1, \omega^{m\ell}a_2, \ldots, \omega^{2\ell}a_m) \quad \text{for} \quad \ell \in \frac{1}{2}\mathbb{Z}. \tag{3.9}$$

Then we have the following lemma, regarding some properties of $G^\ell(\cdot)$.

Lemma 3.5. *For $a \in \mathbb{C}^m$ fixed, and $\ell_1, \ell_2, \ell \in \frac{1}{2}\mathbb{Z}$, $G^{\ell_1}(G^{\ell_2}(a)) = G^{\ell_1+\ell_2}(a)$, and*

$$b_{j,k}(G^\ell(a)) = \omega^{((m+2)k-j)\ell} b_{j,k}(a), \quad \ell \in \frac{1}{2}\mathbb{Z}.$$

In particular,

$$b_j(G^\ell(a)) = \omega^{-j\ell} b_j(a), \ \ \ell \in \mathbb{Z}.$$

Next, recall that the function $f(z, a, \lambda)$ in Theorem 3.2 solves (3.2) and decays to zero exponentially as $z \to \infty$ in S_0, and blows up in $S_{-1} \cup S_1$. One can check that the function

$$f_k(z, a, \lambda) := f(\omega^{-k} z, G^k(a), \omega^{2k}\lambda), \quad k \in \mathbb{Z},$$

which is obtained by scaling $f(z, G^k(a), \omega^{2k}\lambda)$ in the z-variable, also solves (3.2). It is clear that $f_0(z, a, \lambda) = f(z, a, \lambda)$, and that $f_k(z, a, \lambda)$ decays in S_k and blows up in $S_{k-1} \cup S_{k+1}$ since $f(z, G^k(a), \omega^{2k}\lambda)$ decays in S_0. Since no nonconstant solution decays in two consecutive Stokes sectors (see Lemma 3.1 (ii)), f_k and f_{k+1} are linearly independent and hence any solution of (3.2) can be expressed as a linear combination of these two. Especially, for each $k \in \mathbb{Z}$ there exist some coefficients $C_k(a, \lambda)$ and $\widetilde{C}_k(a, \lambda)$ such that

$$f_k(z, a, \lambda) = C_k(a, \lambda) f_0(z, a, \lambda) + \widetilde{C}_k(a, \lambda) f_{-1}(z, a, \lambda). \tag{3.10}$$

We then see that

$$C_k(a, \lambda) = -\frac{\mathcal{W}_{k,-1}(a, \lambda)}{\mathcal{W}_{-1,0}(a, \lambda)} \quad \text{and} \quad \widetilde{C}_k(a, \lambda) = \frac{\mathcal{W}_{k,0}(a, \lambda)}{\mathcal{W}_{-1,0}(a, \lambda)}, \tag{3.11}$$

where $\mathcal{W}_{j,\ell} = f_j f_\ell' - f_j' f_\ell$ is the Wronskian of f_j and f_ℓ. Since both f_j, f_ℓ are solutions of the same linear equation (3.2), we know that the Wronskians are constant functions of z. Also, f_k and f_{k+1} are linearly independent, and hence $\mathcal{W}_{k,k+1} \neq 0$ for all $k \in \mathbb{Z}$.

Also, the following is an easy consequence of (3.10) and (3.11). For each $k, \ell \in \mathbb{Z}$ we have

$$\begin{aligned} \mathcal{W}_{\ell,k}(a, \lambda) &= C_k(a, \lambda)\mathcal{W}_{\ell,0}(a, \lambda) + \widetilde{C}_k(a, \lambda)\mathcal{W}_{\ell,-1}(a, \lambda) \\ &= -\frac{\mathcal{W}_{k,-1}(a, \lambda)\mathcal{W}_{\ell,0}(a, \lambda)}{\mathcal{W}_{-1,0}(a, \lambda)} + \frac{\mathcal{W}_{k,0}(a, \lambda)\mathcal{W}_{\ell,-1}(a, \lambda)}{\mathcal{W}_{-1,0}(a, \lambda)}. \end{aligned} \tag{3.12}$$

Moreover, we have the following lemma that is useful later on.

Lemma 3.6. *Suppose k, $j \in \mathbb{Z}$. Then*

$$\mathcal{W}_{k+1,j+1}(a, \lambda) = \omega^{-1}\mathcal{W}_{k,j}(G(a), \omega^2\lambda), \tag{3.13}$$

and $\mathcal{W}_{0,1}(a, \lambda) = 2\omega^{\mu(a)}$, where $\mu(a) = \frac{m}{4} - \nu(a)$.

Proof. See Sibuya [27, pp. 116-118]. ∎

4. Asymptotics of $\mathcal{W}_{-1,1}(a,\lambda)$

In this section, we introduce asymptotic expansions of $\mathcal{W}_{-1,1}(a,\lambda)$ as $\lambda \to \infty$ along the rays in the complex plane [24].

First, we provide an asymptotic expansion of the Wronskian $\mathcal{W}_{0,j}(a,\lambda)$ of f_0 and f_j that will be frequently used later.

Lemma 4.1. *Suppose that $1 \leq j \leq \frac{m}{2}+1$. Then for each $a \in \mathbb{C}^m$,*

$$\mathcal{W}_{0,j}(a,\lambda) = [2i\omega^{-\frac{j}{2}} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^j(a), \omega^{2j-m-2}\lambda) + L(a,\lambda)\right], \tag{4.1}$$

as $\lambda \to \infty$ in the sector

$$-\pi + \delta \leq \pi - \frac{4j\pi}{m+2} + \delta \leq \arg(\lambda) \leq \pi - \delta. \tag{4.2}$$

Next, we provide an asymptotic expansion of $\mathcal{W}_{-1,1}(a,\lambda)$ as $\lambda \to \infty$ in the sector near the negative real axis.

Theorem 4.2. *Let $m \geq 3$, $a \in \mathbb{C}^m$ and $0 < \delta < \frac{\pi}{m+2}$ be fixed. Then*

$$\mathcal{W}_{-1,1}(a,\lambda) = [2i + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^{-1}(a), \omega^{-2}\lambda) + L(G(a), \omega^{-m}\lambda)\right], \tag{4.3}$$

as $\lambda \to \infty$ along the rays in the sector

$$\pi - \frac{4\pi}{m+2} + \delta \leq \arg(\lambda) \leq \pi + \frac{4\pi}{m+2} - \delta. \tag{4.4}$$

Proof. This is an easy consequence of Lemma 4.1 with $j = 2$ and (3.13). $\square$

Also, for integers $m \geq 4$ we provide an asymptotic expansion of $\mathcal{W}_{-1,1}(a,\lambda)$ as $\lambda \to \infty$ in the sector $|\arg(\lambda)| \leq \pi - \delta$.

Theorem 4.3. *Let $a \in \mathbb{C}^m$ and $0 < \delta < \frac{\pi}{2(m+2)}$ be fixed. If $m \geq 4$ then*

$$\begin{aligned}\mathcal{W}_{-1,1}(a,\lambda) =& [2\omega^{\frac{1}{2}+\mu(a)} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^{-1}(a), \omega^{-2}\lambda) - L(a,\lambda)\right] \\ &+ [2\omega^{\frac{1}{2}+\mu(a)+2\nu(a)} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G(a), \omega^{2}\lambda) - L(a,\lambda)\right],\end{aligned} \tag{4.5}$$

as $\lambda \to \infty$ in the sector

$$-\pi + \delta \leq \arg(\lambda) \leq \pi - \delta. \tag{4.6}$$

Next, we provide an asymptotic expansion of $\mathcal{W}_{-1,1}(a,\lambda)$ as $\lambda \to \infty$ along the rays in the upper- and lower- half planes.

Corollary 4.4. *Let $m \geq 4$, $a \in \mathbb{C}^m$ and $0 < \delta < \frac{\pi}{m+2}$ be fixed. Then*

$$\mathcal{W}_{-1,1}(a,\lambda) = [2\omega^{\frac{1}{2}+\mu(a)} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^{-1}(a), \omega^{-2}\lambda) - L(a,\lambda)\right],$$

as $\lambda \to \infty$ in the sector $\delta \leq \arg(\lambda) \leq \pi - \delta$. Also,

$$\mathcal{W}_{-1,1}(a,\lambda) = [2\omega^{\frac{1}{2}+\mu(a)+2\nu(a)} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G(a), \omega^{2}\lambda) - L(a,\lambda)\right],$$

as $\lambda \to \infty$ in the sector $-\pi + \delta \leq \arg(\lambda) \leq -\delta$.

Proof. We will determine which term in (4.5) dominates in the upper and lower half planes.

Since, by (3.8),

$$\operatorname{Re}\left(L(a,\lambda)\right)=K_m\cos\left(\frac{m+2}{2m}\arg(\lambda)\right)|\lambda|^{\frac{1}{2}+\frac{1}{m}}(1+o(1)),$$

we have

$$\begin{aligned}&\left[\operatorname{Re}\left(L(G^{-1}(a),\omega^{-2}\lambda)\right)-\operatorname{Re}\left(L(a,\lambda)\right)\right]-\left[\operatorname{Re}\left(L(G(a),\omega^{2}\lambda)\right)-\operatorname{Re}\left(L(a,\lambda)\right)\right]\\&=K_m\left[\cos\left(-\frac{2\pi}{m}+\frac{m+2}{2m}\arg(\lambda)\right)-\cos\left(\frac{2\pi}{m}+\frac{m+2}{2m}\arg(\lambda)\right)\right]|\lambda|^{\frac{1}{2}+\frac{1}{m}}(1+o(1))\\&=2K_m\sin\left(\frac{2\pi}{m}\right)\sin\left(\frac{m+2}{2m}\arg(\lambda)\right)|\lambda|^{\frac{1}{2}+\frac{1}{m}}(1+o(1)).\end{aligned}$$

Thus, the first term in (4.5) dominates as $\lambda\to\infty$ along the rays in the upper half plane, and the second term dominates in the lower half plane. This completes the proof. □

Proof of Theorem 4.3. In [24], $C(a,\lambda)$ is used for $\frac{\mathcal{W}_{-1,1}(a,\lambda)}{\mathcal{W}_{0,1}(a,\lambda)}$ and asymptotics of $C(a,\lambda)$ are provided. Notice that $\mathcal{W}_{-1,1}(a,\lambda)=2\omega^{\mu(a)}C(a,\lambda)$.

Theorem 13 in [24] implies (4.5) for the sector

$$\pi-\frac{4\lfloor\frac{m}{2}\rfloor\pi}{m+2}+\delta\le\arg(\lambda)\le\pi-\frac{4\pi}{m+2}-\delta. \tag{4.7}$$

Theorem 14 in [24] implies that

$$\begin{aligned}\mathcal{W}_{-1,1}(a,\lambda)=&[2\omega^{\frac{1}{2}+\mu(a)}+O\left(\lambda^{-\rho}\right)]\exp\left[L(G^{-1}(a),\omega^{-2}\lambda)-L(a,\lambda)\right]\\&+[2\omega^{1+2\mu(a)+4\nu(a)}+O\left(\lambda^{-\rho}\right)]\exp\left[-L(G^{2}(a),\omega^{2-m}\lambda)-L(a,\lambda)\right],\end{aligned}$$

as $\lambda\to\infty$ in the sector $\pi-\frac{8\pi}{m+2}+\delta\le\arg(\lambda)\le\pi-\delta$. One can check that the first term dominates in this sector, by using an argument similar to that in the proof of Corollary 4.4.

Also, Theorem 15 in [24] implies that

$$\begin{aligned}\mathcal{W}_{-1,1}(a,\lambda)=&[2\omega^{1+2\mu(a)}+O\left(\lambda^{-\rho}\right)]\exp\left[-L(a,\omega^{-m-2}\lambda)-L(^{-2}(a),\omega^{-4}\lambda)\right]\\&+[2\omega^{\frac{1}{2}+\mu(a)+2\nu(a)}+O\left(\lambda^{-\rho}\right)]\exp\left[L(G(a),\omega^{-m}\lambda)-L(a,\omega^{-m-2}\lambda)\right],\end{aligned}$$

as $\lambda\to\infty$ in the sector $\pi+\delta\le\arg(\lambda)\le\pi+\frac{8\pi}{m+2}-\delta$. One can check that the second term dominates in this sector. Then we replace λ by $\omega^{m+2}\lambda$ to convert the sector here to $-\pi+\delta\le\arg(\lambda)\le-\pi+\frac{8\pi}{m+2}-\delta$. This completes the proof. □

Theorem 4.5. *Let $m=3$ and let $a\in\mathbb{C}^m$ and $0<\delta<\frac{\pi}{m+2}$ be fixed. Then*

$$\begin{aligned}\mathcal{W}_{-1,1}(a,\lambda)=&[-2\omega^{-\frac{5}{4}}+O\left(\lambda^{-\rho}\right)]\exp\left[L(G^{4}(a),\omega^{-2}\lambda)-L(a,\lambda)\right]\\&-[2i\omega^{\frac{5}{2}}+O\left(\lambda^{-\rho}\right)]\exp\left[-L(G^{2}(a),\omega^{-1}\lambda)-L(a,\lambda)\right],\end{aligned}$$

as $\lambda\to\infty$ in the sector $-\delta\le\arg(\lambda)\le\pi-\delta$. Also,

$$\begin{aligned}\mathcal{W}_{-1,1}(a,\lambda)=&[-2i\omega^{\frac{5}{2}}+O\left(\lambda^{-\rho}\right)]\exp\left[-L(a,\omega^{-5}\lambda)-L(G^{-2}(a),\omega^{-4}\lambda)\right]\\&+[2\omega^{\frac{15}{4}}+O\left(\lambda^{-\rho}\right)]\exp\left[L(G(a),\omega^{-3}\lambda)-L(a,\omega^{-5}\lambda)\right],\end{aligned}$$

as $\lambda \to \infty$ in the sector $-\pi + \delta \le \arg(\lambda) \le \delta$.

Proof. See Theorems 14 and 15 in [24] for a proof. $\square$

5. Asymptotics of $\mathcal{W}_{-1,n}(a, \lambda)$

In this section, we will provide asymptotic expansions of $\mathcal{W}_{-1,n}(a, \cdot)$, zeros of which will be closely related with the eigenvalues of H_n.

First, we treat the cases when $1 \le n < \lfloor \frac{m}{2} \rfloor$.

Theorem 5.1. *Let $1 \le n < \lfloor \frac{m}{2} \rfloor$ be an integer. Then $\mathcal{W}_{-1,n}(a, \cdot)$ admits the following asymptotic expansion*

$$\mathcal{W}_{-1,n}(a, \lambda) = [2\omega^{\frac{2-n}{2}+\mu(G^{n-1}(a))} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^{-1}(a), \omega^{-2}\lambda) - L(G^{n-1}(a), \omega^{2(n-1)}\lambda)\right], \tag{5.1}$$

as $\lambda \to \infty$ in the sector

$$-\frac{2(n-1)\pi}{m+2} + \delta \le \arg(\lambda) \le \pi - \frac{4n\pi}{m+2} + \delta. \tag{5.2}$$

Also,

$$\begin{aligned} \mathcal{W}_{-1,n}(a, \lambda) &= [2\omega^{\frac{2-n}{2}+\mu(G^{n-1}(a))} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^{-1}(a), \omega^{-2}\lambda) - L(G^{n-1}(a), \omega^{2(n-1)}\lambda)\right] \\ &\quad + [2\omega^{\frac{2-n}{2}+\mu(G^{-1}(a))} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^{n}(a), \omega^{2n}\lambda) - L(a, \lambda)\right], \end{aligned} \tag{5.3}$$

as $\lambda \to \infty$ in the sector

$$-\frac{2(n-1)\pi}{m+2} - \delta \le \arg(\lambda) \le -\frac{2(n-1)\pi}{m+2} + \delta. \tag{5.4}$$

Proof. First we will prove (5.1) for the sector

$$-\frac{2(n-1)\pi}{m+2} + \delta \le \arg(\lambda) \le \pi - \frac{4n\pi}{m+2} - \delta \tag{5.5}$$

and the second part of the theorem by induction on n.

The case when $n = 1$ is trivially satisfied by Theorem 4.3 and Corollary 4.4 since $\mu(a) + 2\nu(a) = \mu(G^{-1}(a))$.

Suppose that (5.1) holds in the sector (5.5) for $n - 1$. From this induction hypothesis we have

$$\begin{aligned} \mathcal{W}_{0,n}(a, \lambda) &= \omega^{-1}\mathcal{W}_{-1,n-1}(G(a), \omega^2\lambda) \\ &= [2\omega^{-\frac{n-1}{2}+\mu(G^{n-1}(a))} + O\left(\lambda^{-\rho}\right)] \exp\left[L(a, \lambda) - L(G^{n-1}(a), \omega^{2(n-1)}\lambda)\right], \end{aligned} \tag{5.6}$$

as $\lambda \to \infty$ in the sector

$$-\frac{2(n-2)\pi}{m+2} + \delta \le \arg(\omega^2\lambda) \le \pi - \frac{4(n-1)\pi}{m+2} - \delta,$$

that is,

$$-\frac{2n\pi}{m+2} + \delta \le \arg(\lambda) \le \pi - \frac{4n\pi}{m+2} - \delta. \tag{5.7}$$

Also, from Lemma 4.1 if $1 \le j \le \frac{m}{2}+1$, then we have

$$\mathcal{W}_{0,j}(a,\lambda) = [2i\omega^{-\frac{j}{2}} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^j(a), \omega^{2j-m-2}\lambda) + L(a,\lambda)\right], \tag{5.8}$$

as $\lambda \to \infty$ in the sector

$$\pi - \frac{4j\pi}{m+2} + \delta \le \arg(\lambda) \le \pi - \delta.$$

We solve (3.12) for $\mathcal{W}_{\ell,-1}(a,\lambda)$ and set $\ell = n$ to get

$$\mathcal{W}_{-1,n}(a,\lambda) = \frac{\mathcal{W}_{-1,0}(a,\lambda)\mathcal{W}_{n,k}(a,\lambda)}{\mathcal{W}_{0,k}(a,\lambda)} + \frac{\mathcal{W}_{0,n}(a,\lambda)\mathcal{W}_{-1,k}(a,\lambda)}{\mathcal{W}_{0,k}(a,\lambda)} \tag{5.9}$$

Set $k = \lfloor \frac{m}{2} \rfloor$. Then since $1 \le k-n < k = \lfloor \frac{m}{2} \rfloor$, using (3.13),

$$\begin{aligned}
\mathcal{W}_{-1,n}(a,\lambda) &= \frac{2\omega^{\mu(G^{-1}(a))}\mathcal{W}_{0,k-n}(G^n(a),\omega^{2n}\lambda)}{\omega^{n-1}\mathcal{W}_{0,k}(a,\lambda)} + \frac{\mathcal{W}_{0,n}(a,\lambda)\mathcal{W}_{0,k+1}(G^{-1}(a),\omega^{-2}\lambda)}{\omega^{-1}\mathcal{W}_{0,k}(a,\lambda)} \\
&= \frac{2\omega^{\mu(G^{-1}(a))}[2i\omega^{-\frac{k-n}{2}} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^n(a),\omega^{2k-m-2}\lambda) + L(G^n(a),\omega^{2n}\lambda)\right]}{\omega^{n-1}[2i\omega^{-\frac{k}{2}} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^k(a),\omega^{2k-m-2}\lambda) + L(a,\lambda)\right]} \\
&+ \frac{[2\omega^{-\frac{n-1}{2}+\mu(G^{n-1}(a))} + O\left(\lambda^{-\rho}\right)] \exp\left[L(a,\lambda) - L(G^{n-1}(a),\omega^{2(n-1)}\lambda)\right]}{\omega^{-1}[2i\omega^{-\frac{k}{2}} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^k(a),\omega^{2k-m-2}\lambda) + L(a,\lambda)\right]} \\
&\times [2i\omega^{-\frac{k+1}{2}} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^k(a),\omega^{2k-m-2}\lambda) + L(G^{-1}(a),\omega^{-2}\lambda)\right] \\
&= [2\omega^{\frac{2-n}{2}+\mu(G^{-1}(a))} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^n(a),\omega^{2n}\lambda) - L(a,\lambda)\right] \\
&+ [2\omega^{\frac{2-n}{2}+\mu(G^{n-1}(a))} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^{-1}(a),\omega^{-2}\lambda) - L(G^{n-1}(a),\omega^{2(n-1)}\lambda)\right],
\end{aligned} \tag{5.10}$$

where we used (5.6) for $\mathcal{W}_{0,n}(a,\lambda)$ and (5.8) for everything else, provided that λ lies in (5.7) and that

$$\begin{aligned}
\pi - \frac{4\left(\lfloor \frac{m}{2} \rfloor - n\right)\pi}{m+2} + \delta \le \arg(\omega^{2n}\lambda) \le \pi - \delta \\
\pi - \frac{4\lfloor \frac{m}{2} \rfloor \pi}{m+2} + \delta \le \arg(\lambda) \le \pi - \delta \\
\pi - \frac{4\left(\lfloor \frac{m}{2} \rfloor + 1\right)\pi}{m+2} + \delta \le \arg(\omega^{-2}\lambda) \le \pi - \delta,
\end{aligned}$$

that is,

$$-\frac{2n\pi}{m+2} + \delta \le \arg(\lambda) \le \pi - \frac{4n\pi}{m+2} - \delta.$$

Thus, the second part of the theorem is proved by induction.

Next in order to prove the first part of the theorem for the sector (5.5), we will determine which term in (5.10) dominates as $\lambda \to \infty$. To do that, we look at

$$\begin{aligned}
&\mathrm{Re}\,\big(L(G^{-1}(a),\omega^{-2}\lambda) - L(G^{n-1}(a),\omega^{2(n-1)}\lambda)\big) - \mathrm{Re}\,\big(L(G^{n}(a),\omega^{2n}\lambda) - L(a,\lambda)\big)\\
&= K_m\left[\cos\left(-\frac{2\pi}{m} + \frac{m+2}{2m}\arg(\lambda)\right) - \cos\left(\frac{2(n-1)\pi}{m} + \frac{m+2}{2m}\arg(\lambda)\right)\right.\\
&\left.- \left(\cos\left(-\frac{2n\pi}{m} + \frac{m+2}{2m}\arg(\lambda)\right) - \cos\left(\frac{m+2}{2m}\arg(\lambda)\right)\right)\right] |\lambda|^{\frac{1}{2}+\frac{1}{m}}(1+o(1))\\
&= 2K_m \sin\left(\frac{n\pi}{m}\right)\left[\sin\left(\frac{(n-2)\pi}{m} + \frac{m+2}{2m}\arg(\lambda)\right)\right.\\
&\qquad\qquad\left. + \sin\left(\frac{n\pi}{m} + \frac{m+2}{2m}\arg(\lambda)\right)\right] |\lambda|^{\frac{1}{2}+\frac{1}{m}}(1+o(1))\\
&= 4K_m \sin\left(\frac{n\pi}{m}\right)\cos\left(\frac{\pi}{m}\right)\sin\left(\frac{(n-1)\pi}{m} + \frac{m+2}{2m}\arg(\lambda)\right) |\lambda|^{\frac{1}{2}+\frac{1}{m}}(1+o(1)), \qquad (5.11)
\end{aligned}$$

that tends to positive infinity as $\lambda \to \infty$ (and hence the second term in (5.10) dominates) if $-\frac{2(n-1)\pi}{m+2} + \delta \le \arg(\lambda) \le \pi - \frac{4n\pi}{m+2} - \delta$.

We still need to prove (5.1) for the sector

$$\pi - \frac{4n\pi}{m+2} - \delta \le \arg(\lambda) \le \pi - \frac{4n\pi}{m+2} + \delta, \qquad (5.12)$$

for which we use induction on n again.

When $n = 1$, (5.1) holds by Lemma 4.4.

Suppose that (5.1) in the sector (5.12) for $n-1$ with $2 \le n < \lfloor \frac{m}{2} \rfloor$. Then (3.13) and (5.9) with $k = n+1$ yield

$$\mathcal{W}_{-1,n}(a,\lambda) = \frac{2\omega^{\mu(G^{-1}(a))}\mathcal{W}_{0,1}(G^n(a),\omega^{2n}\lambda)}{\omega^{n-1}\mathcal{W}_{0,n+1}(a,\lambda)} + \frac{\mathcal{W}_{-1,n-1}(G(a),\omega^2\lambda)\mathcal{W}_{0,n+2}(G^{-1}(a),\omega^{-2}\lambda)}{\mathcal{W}_{0,n+1}(a,\lambda)}.$$

If $\pi - \frac{4n\pi}{m+2} - \delta \le \arg(\lambda) \le \pi - \frac{4n\pi}{m+2} + \delta$, then $\pi - \frac{4(n-1)\pi}{m+2} - \delta \le \arg(\omega^2\lambda) \le \pi - \frac{4(n-1)\pi}{m+2} + \delta$. So

(5.13)

$$\begin{aligned}
\mathcal{W}_{-1,n}(a,\lambda) &= \frac{4\omega^{\mu(G^{-1}(a))+\mu(G^n(a))}}{\omega^{n-1}\mathcal{W}_{0,n+1}(a,\lambda)} + \frac{\mathcal{W}_{-1,n-1}(G(a),\omega^2\lambda)\mathcal{W}_{0,n+2}(G^{-1}(a),\omega^{-2}\lambda)}{\mathcal{W}_{0,n+1}(a,\lambda)}\\
&= \frac{4\omega^{\mu(G^{-1}(a))+\mu(G^n(a))}}{\omega^{n-1}[2i\omega^{-\frac{n+1}{2}} + O(\lambda^{-\rho})]\exp[L(G^{n+1}(a),\omega^{2n-m}\lambda) + L(a,\lambda)]}\\
&\quad + \frac{[2\omega^{\frac{3-n}{2}+\mu(G^{n-1}(a))} + O(\lambda^{-\rho})]\exp\left[L(a,\lambda) - L(G^{n-1}(a),\omega^{2(n-1)}\lambda)\right]}{[2i\omega^{-\frac{n+1}{2}} + O(\lambda^{-\rho})]\exp[L(G^{n+1}(a),\omega^{2n-m}\lambda) + L(a,\lambda)]}\\
&\quad \times [2i\omega^{-\frac{n+2}{2}} + O(\lambda^{-\rho})]\exp\left[L(G^{n+1}(a),\omega^{2n-m}\lambda) + L(G^{-1}(a),\omega^{-2}\lambda)\right]\\
&= [-2i\omega^{\frac{3-n}{2}+\mu(G^{-1}(a))+\mu(G^n(a))} + O(\lambda^{-\rho})]\exp\left[-L(G^{n+1}(a),\omega^{2n-m}\lambda) - L(a,\lambda)\right]\\
&\quad + [2\omega^{\frac{2-n}{2}+\mu(G^{n-1}(a))} + O(\lambda^{-\rho})]\exp\left[L(G^{-1}(a),\omega^{-2}\lambda) - L(G^{n-1}(a),\omega^{2(n-1)}\lambda)\right],
\end{aligned}$$

where we use the induction hypothesis for $\mathcal{W}_{-1,n-1}(G(a),\omega^2\lambda)$ and use (4.1) for $\mathcal{W}_{0,n+1}(a,\lambda)$ and $\mathcal{W}_{0,n+2}(G^{-1}(a),\omega^{-2}\lambda)$. Next, we use an argument similar (5.11) to complete the induction step. Thus, the theorem is proved. $\square$

Next we investigate $\mathcal{W}_{0,\lfloor\frac{m}{2}\rfloor}(a,\lambda)$.

Theorem 5.2. *If $m\geq 4$ is an even integer, then*

$$\begin{aligned}&\mathcal{W}_{-1,\lfloor\frac{m}{2}\rfloor}(a,\lambda)\\ &= -[2\omega^{2+\mu(G^{-1}(a))+\mu(G^{\frac{m}{2}}(a))}+O\left(\lambda^{-\rho}\right)]\exp\left[-L(G^{\frac{m}{2}+1}(a),\lambda)-L(a,\lambda)\right]\\ &\quad-[2\omega^{2+\mu(a)+\mu(G^{\frac{m-2}{2}}(a))}+O\left(\lambda^{-\rho}\right)]\exp\left[-L(G^{\frac{m-2}{2}}(a),\omega^{m-2}\lambda)-L(G^{m}(a),\omega^{m-2}\lambda)\right].\end{aligned}\tag{5.14}$$

as $\lambda\to\infty$ in the sector

$$-\pi+\frac{4\pi}{m+2}-\delta\leq\arg(\lambda)\leq-\pi+\frac{4\pi}{m+2}+\delta.\tag{5.15}$$

If $m\geq 4$ is an odd integer, then

$$\begin{aligned}\mathcal{W}_{-1,\lfloor\frac{m}{2}\rfloor}(a,\lambda)&=[2\omega^{\frac{5}{4}}+O\left(\lambda^{-\rho}\right)]\exp\left[-L(G^{\frac{m+1}{2}}(a),\omega^{-1}\lambda)-L(a,\lambda)\right]\\ &\quad+[2\omega^{\frac{5}{4}}+O\left(\lambda^{-\rho}\right)]\exp\left[L(G^{m+1}(a),\omega^{-2}\lambda)-L(G^{\frac{m-3}{2}}(a),\omega^{m-3}\lambda)\right].\end{aligned}\tag{5.16}$$

as $\lambda\to\infty$ in the sector

$$-\pi+\frac{4\pi}{m+2}+\delta\leq\arg(\lambda)\leq-\pi+\frac{6\pi}{m+2}+\delta.$$

Proof. We will use (5.13) with $n=\lfloor\frac{m}{2}\rfloor$, that is,

$$\mathcal{W}_{-1,\lfloor\frac{m}{2}\rfloor}(a,\lambda)=\frac{4\omega^{\mu(G^{-1}(a))+\mu(G^{\lfloor\frac{m}{2}\rfloor}(a))}}{\omega^{\lfloor\frac{m}{2}\rfloor-1}\mathcal{W}_{0,\lfloor\frac{m}{2}\rfloor+1}(a,\lambda)}+\frac{\mathcal{W}_{-1,\lfloor\frac{m}{2}\rfloor-1}(G(a),\omega^2\lambda)\mathcal{W}_{0,\lfloor\frac{m}{2}\rfloor+2}(G^{-1}(a),\omega^{-2}\lambda)}{\mathcal{W}_{0,\lfloor\frac{m}{2}\rfloor+1}(a,\lambda)}.$$

When m is even, say $m=2k$,

$$\begin{aligned}\mathcal{W}_{0,k+2}(G^{-1}(a),\omega^{-2}\lambda)&=\mathcal{W}_{m+2,k+2}(G^{-1}(a),\omega^{-2}\lambda)\\ &=-\omega^{-k-3}\mathcal{W}_{-1,k-1}(G^{k+2}(a),\omega^{2k+4}\lambda)\\ &=\omega^{-2}\mathcal{W}_{-1,k-1}(G^{k+2}(a),\omega^{2}\lambda).\end{aligned}$$

So

$$\begin{aligned}\mathcal{W}_{-1,k}(a,\lambda)&=\frac{4\omega^{\mu(G^{-1}(a))+\mu(G^{k}(a))}}{\omega^{k-1}\mathcal{W}_{0,k+1}(a,\lambda)}+\frac{\mathcal{W}_{-1,k-1}(G(a),\omega^2\lambda)\mathcal{W}_{0,k+2}(G^{-1}(a),\omega^{-2}\lambda)}{\mathcal{W}_{0,k+1}(a,\lambda)}\\ &=\frac{4\omega^{\mu(G^{-1}(a))+\mu(G^{k}(a))}}{\omega^{k-1}\mathcal{W}_{0,k+1}(a,\lambda)}+\frac{\mathcal{W}_{-1,k-1}(G(a),\omega^2\lambda)\mathcal{W}_{-1,k-1}(G^{k+2}(a),\omega^{2}\lambda)}{\omega^2\mathcal{W}_{0,k+1}(a,\lambda)}\end{aligned}$$

Since λ lies in (5.15),

$$-\frac{2\left(\lfloor\frac{m}{2}\rfloor-2\right)\pi}{m+2}+\delta\leq-\pi+\frac{8\pi}{m+2}-\delta\leq\arg(\omega^2\lambda)\leq\pi-\frac{4\left(\lfloor\frac{m}{2}\rfloor-1\right)\pi}{m+2}+\delta.$$

$$
\begin{aligned}
&\mathcal{W}_{-1,k}(a,\lambda)\\
&=\frac{4\omega^{\mu(G^{-1}(a))+\mu(G^k(a))}}{\omega^{k-1}\mathcal{W}_{0,k+1}(a,\lambda)}+\frac{\mathcal{W}_{-1,k-1}(G(a),\omega^2\lambda)\mathcal{W}_{0,k+2}(G^{-1}(a),\omega^{-2}\lambda)}{\mathcal{W}_{0,k+1}(a,\lambda)}\\
&=\frac{4\omega^{\mu(G^{-1}(a))+\mu(G^k(a))}}{\omega^{k-1}\mathcal{W}_{0,k+1}(a,\lambda)}+\frac{\mathcal{W}_{-1,k-1}(G(a),\omega^2\lambda)\mathcal{W}_{-1,k-1}(G^{k+2}(a),\omega^2\lambda)}{\omega^2\mathcal{W}_{0,k+1}(a,\lambda)}\\
&=\frac{4\omega^{\mu(G^{-1}(a))+\mu(G^k(a))}}{\omega^{k-1}[2i\omega^{-\frac{k+1}{2}}+O\left(\lambda^{-\rho}\right)]\exp\left[L(G^{k+1}(a),\omega^{2k-m}\lambda)+L(a,\lambda)\right]}\\
&+\frac{[2\omega^{\frac{3-k}{2}+\mu(G^{k-1}(a))}+O\left(\lambda^{-\rho}\right)]\exp\left[L(a,\lambda)-L(G^{k-1}(a),\omega^{2(k-1)}\lambda)\right]}{\omega^2[2i\omega^{-\frac{k+1}{2}}+O\left(\lambda^{-\rho}\right)]\exp\left[L(G^{k+1}(a),\omega^{2k-m}\lambda)+L(a,\lambda)\right]}\\
&\times[2\omega^{\frac{3-k}{2}+\mu(G^{2k}(a))}+O\left(\lambda^{-\rho}\right)]\exp\left[L(G^{k+1}(a),\lambda)-L(G^{2k}(a),\omega^{2(k-1)}\lambda)\right]\\
&=[-2i\omega^{\frac{3-k}{2}+\mu(G^{-1}(a))+\mu(G^k(a))}+O\left(\lambda^{-\rho}\right)]\exp\left[-L(G^{k+1}(a),\lambda)-L(a,\lambda)\right]\\
&-[2i\omega^{\frac{3-k}{2}+\mu(G^{k-1}(a))+\mu(G^{2k}(a))}+O\left(\lambda^{-\rho}\right)]\exp\left[-L(G^{k-1}(a),\omega^{2(k-1)}\lambda)-L(G^{2k}(a),\omega^{2(k-1)}\lambda)\right]\\
&=-[2i\omega^{\frac{3-k}{2}+\mu(G^{-1}(a))+\mu(G^k(a))}+O\left(\lambda^{-\rho}\right)]\exp\left[-L(G^{k+1}(a),\lambda)-L(a,\lambda)\right]\\
&-[2i\omega^{\frac{3-k}{2}+\mu(a)+\mu(G^{k-1}(a))}+O\left(\lambda^{-\rho}\right)]\exp\left[-L(G^{k-1}(a),\omega^{2(k-1)}\lambda)-L(G^{2k}(a),\omega^{2(k-1)}\lambda)\right],
\end{aligned}
$$

where we used (5.1) for $\mathcal{W}_{-1,k-1}(G(a),\cdot)$ and $\mathcal{W}_{-1,k-1}(G^{k+2}(a),\cdot)$, and (4.1) for $\mathcal{W}_{0,k+1}(a,\cdot)$. Finally, we use $\omega^{-\frac{m+2}{4}}=-i$, to get the desired asymptotic expansion of $\mathcal{W}_{-1,\lfloor\frac{m}{2}\rfloor}(a,\lambda)$.

Next we investigate the case when m is odd, say $m=2k+1$ (so $\lfloor\frac{m}{2}\rfloor=k$).

$$\mathcal{W}_{0,\lfloor\frac{m}{2}\rfloor+1}(a,\lambda)=\omega^{-1}\mathcal{W}_{-1,\lfloor\frac{m}{2}\rfloor}(G(a),\omega^2\lambda)$$

and

$$
\begin{aligned}
\mathcal{W}_{0,k+2}(G^{-1}(a),\omega^{-2}\lambda)&=\mathcal{W}_{m+2,k+2}(G^{-1}(a),\omega^{-2}\lambda)\\
&=-\omega^{-k-2}\mathcal{W}_{0,k+1}(G^{k+1}(a),\omega^{2k+2}\lambda)\\
&=\omega^{-\frac{1}{2}}\mathcal{W}_{0,k+1}(G^{k+1}(a),\omega^{-1}\lambda).
\end{aligned}
$$

Similarly to the proof of the theorem for m even,

$$
\begin{aligned}
\mathcal{W}_{-1,k}(a,\lambda)&=\frac{4\omega^{\mu(G^{-1}(a))+\mu(G^k(a))}}{\omega^{k-1}\mathcal{W}_{0,k+1}(a,\lambda)}+\frac{\mathcal{W}_{-1,k-1}(G(a),\omega^2\lambda)\mathcal{W}_{0,k+2}(G^{-1}(a),\omega^{-2}\lambda)}{\mathcal{W}_{0,k+1}(a,\lambda)}\\
&=\frac{4\omega^{\mu(G^{-1}(a))+\mu(G^k(a))}}{\omega^{k-1}\mathcal{W}_{0,k+1}(a,\lambda)}+\frac{\mathcal{W}_{-1,k-1}(G(a),\omega^2\lambda)\mathcal{W}_{0,k+1}(G^{k+1}(a),\omega^{-1}\lambda)}{\omega^{\frac{1}{2}}\mathcal{W}_{0,k+1}(a,\lambda)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4\omega^{\mu(G^{-1}(a))+\mu(G^k(a))}}{\omega^{k-1}[2i\omega^{-\frac{k+1}{2}} + O(\lambda^{-\rho})] \exp[L(G^{k+1}(a), \omega^{2k-m}\lambda) + L(a,\lambda)]} \\
&+ \frac{[2\omega^{\frac{3-k}{2}+\mu(G^{k-1}(a))} + O(\lambda^{-\rho})] \exp\left[L(a,\lambda) - L(G^{k-1}(a), \omega^{2(k-1)}\lambda)\right]}{\omega^{\frac{1}{2}}[2i\omega^{-\frac{k+1}{2}} + O(\lambda^{-\rho})] \exp[L(G^{k+1}(a), \omega^{2k-m}\lambda) + L(a,\lambda)]} \\
&\times [2i\omega^{-\frac{k+1}{2}} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^{2k+2}(a), \omega^{2k-m-1}\lambda) + L(G^{k+1}(a), \omega^{-1}\lambda)\right] \\
&= [-2i\omega^{\frac{3-k}{2}+\mu(G^{-1}(a))+\mu(G^k(a))} + O\left(\lambda^{-\rho}\right)] \exp\left[-L(G^{k+1}(a), \omega^{-1}\lambda) - L(a,\lambda)\right] \\
&+ [2\omega^{\frac{2-k}{2}+\mu(G^{k-1}(a))} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^{2k+2}(a), \omega^{-2}\lambda) - L(G^{k-1}(a), \omega^{2(k-1)}\lambda)\right] \\
&= -[2i\omega^{\frac{3-k}{2}+\frac{m}{2}} + O\left(\lambda^{-\rho}\right)] \exp\left[-L(G^{k+1}(a), \omega^{-1}\lambda) - L(a,\lambda)\right] \\
&+ [2\omega^{\frac{2-k}{2}+\frac{m}{4}} + O\left(\lambda^{-\rho}\right)] \exp\left[L(G^{2k+2}(a), \omega^{-2}\lambda) - L(G^{k-1}(a), \omega^{2(k-1)}\lambda)\right],
\end{aligned}
$$

where we use (5.1) for $\mathcal{W}_{-1,k-1}(G(a),\cdot)$, and use (4.1) for $\mathcal{W}_{0,k+1}(a,\cdot)$ and $\mathcal{W}_{0,k+1}(G^{k+1}(a),\cdot)$. Finally, we use $\omega^{\frac{m+2}{4}} = i$, to get the asymptotic expansion of $\mathcal{W}_{-1,\lfloor\frac{m}{2}\rfloor}(a,\lambda)$. This completes the proof. $\square$

The *order of an entire function* g is defined by

$$\limsup_{r\to\infty} \frac{\log\log M(r,g)}{\log r},$$

where $M(r,g) = \max\{|g(re^{i\theta})| : 0 \le \theta \le 2\pi\}$ for $r > 0$. If for some positive real numbers σ, c_1, c_2, we have $\exp[c_1 r^\sigma] \le M(r,g) \le \exp[c_2 r^\sigma]$ for all large r, then the order of g is σ.

Corollary 5.3. *Let $1 \le n \le \frac{m}{2}$. Then the entire functions $\mathcal{W}_{-1,n}(a,\cdot)$ are of order $\frac{1}{2} + \frac{1}{m}$, and hence they have infinitely many zeros in the complex plane. Moreover, $\mathcal{W}_{-1,n}(a,\cdot)$ have at most finitely many zeros in the sector*

$$-\frac{2(n-1)\pi}{m+2} + \delta \le \arg(\lambda) \le \pi + \frac{4\pi}{m+2} - \delta.$$

6. Relation between eigenvalues of H_ℓ and zeros of $\mathcal{W}_{-1,n}(a,\cdot)$

In this section, we will relate the eigenvalues of H_ℓ with zeros of some entire function $\mathcal{W}_{-1,n}(a,\cdot)$.

Suppose that $\ell = 2s-1$ is odd with $1 \le \ell \le m-1$. Then (1.1) becomes (3.2) by the scaling $v(z) = u(-iz)$, and v decays in the Stokes sectors S_{-s} and S_s. Since f_{s-1} and f_s are linearly independent, for some D_s and $\widetilde{D}_s$ one can write

$$f_{-s}(z,a,\lambda) = D_s(a,\lambda) f_{s-1}(z,a,\lambda) + \widetilde{D}_s(a,\lambda) f_s(z,a,\lambda).$$

Then one finds

$$D_s(a,\lambda) = \frac{\mathcal{W}_{-s,s}(a,\lambda)}{\mathcal{W}_{s-1,s}(a,\lambda)} \quad \text{and} \quad \widetilde{D}_s(a,\lambda) = \frac{\mathcal{W}_{-s,s-1}(a,\lambda)}{\mathcal{W}_{s,s-1}(a,\lambda)}.$$

Also it is easy to see that λ is an eigenvalue of H_ℓ if and only if $D_s(a,\lambda) = 0$ if and only if $\mathcal{W}_{-s,s}(a,\lambda) = 0$. Since $\mathcal{W}_{-s,s}(a,\lambda) = \omega^{s-1}\mathcal{W}_{-1,2s-1}(G^{-s+1}(a), \omega^{-2s+2}\lambda)$, by Corollary 5.3

$\mathcal{W}_{-s,s}(a,\lambda)$ has at most finitely many zeros in the sector $-\frac{2(2s-2)\pi}{m+2}+\delta \le \arg(\omega^{-2s+2}\lambda) \le \pi + \frac{4\pi}{m+2} - \delta$, that is,

$$\delta \le \arg(\lambda) \le \pi + \frac{2(2s+1)\pi}{m+2} - \delta.$$

Next, by symmetry one can show that $\mathcal{W}_{-s,s}(a,\lambda)$ has at most finitely many zeros in the sector $\pi \le \arg(\lambda) \le 2\pi - \delta$. For that, we examine H_ℓ with $P(z)$ replaced by $\overline{P(\overline{z})}$ whose coefficient vector is $\overline{a} := (\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_m)$. Then one sees that $\mathcal{W}_{-s,s}(a,\lambda) = 0$ if and only if $\mathcal{W}_{-s,s}(\overline{a},\overline{\lambda}) = 0$. $\mathcal{W}_{-s,s}(\overline{a},\overline{\lambda})$ has at most finitely many zeros in the sector $\delta \le \arg(\overline{\lambda}) \le \pi$ by the arguments above. Thus, $\mathcal{W}_{-s,s}(a,\lambda)$ has at most finitely many zeros in the sector $\pi \le \arg(\lambda) \le 2\pi - \delta$, and has infinitely many zeros in the sector $|\arg(\lambda)| \le \delta$ since it is an entire function of order $\frac{1}{2} + \frac{1}{m} \in (0,1)$.

Suppose that $\ell = 2s$ is even with $1 \le \ell \le m-1$. Then (1.1) becomes (3.4) by the scaling $y(z) = u(-i\omega^{-\frac{1}{2}}z)$, and y decays in the Stokes sectors S_{-s} and S_{s+1}. We then see that the coefficient vector $\widetilde{a}$ of the polynomial $\omega^{-1}P(\omega^{-\frac{1}{2}}z)$ becomes

$$\widetilde{a} = G^{-\frac{1}{2}}(a).$$

Now one can express f_{-s} as a linear combination of f_s and f_{s+1} as follows.

$$f_{-s}(z,\widetilde{a},\omega^{-1}\lambda) = \frac{\mathcal{W}_{-s,s+1}(\widetilde{a},\omega^{-1}\lambda)}{\mathcal{W}_{s,s+1}(\widetilde{a},\omega^{-1}\lambda)} f_s(z,\widetilde{a},\omega^{-1}\lambda) + \frac{\mathcal{W}_{-s,s}(\widetilde{a},\omega^{-1}\lambda)}{\mathcal{W}_{s+1,s}(\widetilde{a},\omega^{-1}\lambda)} f_{s+1}(z,\widetilde{a},\omega^{-1}\lambda).$$

Thus, λ is an eigenvalue of H_ℓ if and only if $\mathcal{W}_{-s,s+1}(\widetilde{a},\omega^{-1}\lambda) = 0$. Since

$$\mathcal{W}_{-s,s+1}(\widetilde{a},\omega^{-1}\lambda) = \omega^{s-1}\mathcal{W}_{-1,2s}(G^{-s+1}(\widetilde{a}),\omega^{-2s+1}\lambda),$$

by Corollary 5.3, $\mathcal{W}_{-s,s+1}(\widetilde{a},\omega^{-1}\lambda)$ has at most finitely many zeros in the sector $-\frac{2(2s-1)\pi}{m+2} + \delta \le \arg(\omega^{-2s+1}\lambda) \le \pi + \frac{4\pi}{m+2} - \delta$, that is,

$$\delta \le \arg(\lambda) \le \pi + \frac{4s\pi}{m+2} - \delta.$$

This is true for each $a \in \mathbb{C}^m$. So one can show that $\mathcal{W}_{-s,s+1}(\widetilde{a},\omega^{-1}\lambda)$ has at most finitely many zeros in the sector $\pi \le \arg(\lambda) \le 2\pi - \delta$ by symmetry, similar to the case when ℓ is odd. Thus, $\mathcal{W}_{-s,s+1}(a,\lambda)$ has infinitely many zeros in the sector $|\arg(\lambda)| \le \delta$.

7. Proof of Theorem 1.1 when $1 \le \ell < \lfloor \frac{m}{2} \rfloor$

In this section, we prove Theorem 1.1 for $1 \le \ell < \lfloor \frac{m}{2} \rfloor$ and in doing so, we will use the following proposition on univalent functions.

Proposition 7.1 ([17, p. 216]). *Let $A(\mu)$ be analytic in the region $S = \{\mu \in \mathbb{C} : \alpha \le \arg(\mu) \le \beta, |\mu| \ge M\}$ for some $\alpha, \beta \in \mathbb{R}$ with $\beta - \alpha < \pi$ and for some $M > 0$. Suppose that $A(\mu) \to 0$ as $\mu \to \infty$ in S. Then there exist $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{M}$ such that $\mu(1 + A(\mu))$ is univalent in $\{\mu \in \mathbb{C} : \widetilde{\alpha} \le \arg(\mu) \le \widetilde{\beta}, |\mu| \ge \widetilde{M}\} \subset S$.*

Proof. See [17, p. 216] or [27, Thm. 3.4] for a proof. □

We will consider two cases; when ℓ is odd and when ℓ is even.

Proof of Theorem 1.1 when $1 \leq \ell < \lfloor \frac{m}{2} \rfloor$ is odd. We will closely follow his proof of Theorem 29.1 in [27] where Sibuya computed the leading term in the asymptotics (1.3) for $\ell = 1$.

Suppose that $1 \leq \ell = 2s-1 < \lfloor \frac{m}{2} \rfloor$ is odd. Recall that when ℓ is odd, λ is an eigenvalue of H_ℓ if and only if $\mathcal{W}_{-s,s}(a,\lambda) = 0$. Also, in Section 6 we showed that $\mathcal{W}_{-s,s}(a,\lambda)$ has all zeros except finitely many in the sector $|\arg(\lambda)| \leq \delta$ and hence, all the eigenvalues λ of H_ℓ lie in the sector $|\arg(\lambda)| \leq \delta$ if $|\lambda|$ is large enough.

Since

$$\mathcal{W}_{-s,s}(a,\lambda) = \omega^{s-1}\mathcal{W}_{-1,2s-1}(G^{-s+1}(a), \omega^{-2s+2}\lambda),$$

we will use (5.3) to investigate asymptotics of large eigenvalues. Suppose that $\mathcal{W}_{-s,s}(a,\lambda) = 0$ and $|\lambda|$ is large enough. Then from (5.3) with $n = 2s-1$, and with a and λ replaced by $G^{-s+1}(a)$ and $\omega^{-2s+2}\lambda$, respectively, we have

$$\begin{aligned}&\left[1 + O\left(\lambda^{-\rho}\right)\right] \exp\left[L(G^s(a), \omega^{2s}\lambda) - L(G^{-s}(a), \omega^{-2s}\lambda)\right]\\ &\qquad\times \exp\left[L(G^{s-1}(a), \omega^{2s-2}\lambda) - L(G^{-s+1}(a), \omega^{-2s+2}\lambda)\right] = -\omega^{2\nu(G^s(a))}.\end{aligned}$$

Also, since $[1 + O(\lambda^{-\rho})] = \exp[O(\lambda^{-\rho})]$ and $\omega^{2\nu(G^s(a))} = \exp\left[\frac{4\pi\nu(G^s(a))}{m+2}\right]$,

$$\begin{aligned}&\exp\left[L(G^s(a), \omega^{2s}\lambda) - L(G^{-s}(a), \omega^{-2s}\lambda)\right]\\ &\times \exp\left[L(G^{s-1}(a), \omega^{2s-2}\lambda) - L(G^{-s+1}(a), \omega^{-2s+2}\lambda) - \frac{4\pi\nu(G^s(a))}{m+2} + O\left(\lambda^{-\rho}\right)\right] = -1.\end{aligned}\tag{7.1}$$

For each odd integer $\ell = 2s-1$ in $1 \leq \ell < \lfloor \frac{m}{2} \rfloor$, we define

$$\begin{aligned}h_{m,\ell}(\lambda) =& L(G^s(a), \omega^{2s}\lambda) - L(G^{-s}(a), \omega^{-2s}\lambda)\\ &+ L(G^{s-1}(a), \omega^{2s-2}\lambda) - L(G^{-s+1}(a), \omega^{-2s+2}\lambda) - \frac{4\pi\nu(G^s(a))}{m+2} + O\left(\lambda^{-\rho}\right).\end{aligned}$$

Then by Corollary 3.4,

$$\begin{aligned}h_{m,\ell}(\lambda) &= K_m\left(e^{\frac{2s\pi}{m}i} - e^{-\frac{2s\pi}{m}i} + e^{\frac{2(s-1)\pi}{m}i} - e^{-\frac{2(s-1)\pi}{m}i}\right)\lambda^{\frac{m+2}{2m}}(1+o(1))\\ &= 2iK_m\left(\sin\left(\frac{2s\pi}{m}\right) + \sin\left(\frac{2s\pi}{m} - \frac{2\pi}{m}\right)\right)\lambda^{\frac{m+2}{2m}}(1+o(1))\\ &= 4iK_m\cos\left(\frac{\pi}{m}\right)\sin\left(\frac{(2s-1)\pi}{m}\right)\lambda^{\frac{m+2}{2m}}(1+o(1)) \quad \text{as} \quad \lambda \to \infty,\end{aligned}\tag{7.2}$$

in the sector $|\arg(\lambda)| \leq \delta$. Since $K_m > 0$ and $0 < \frac{(2s-1)\pi}{m} < \pi$, the function $h_{m,\ell}(\cdot)$ maps the region $|\lambda| \geq M_1$ for some large M_1 and $|\arg(\lambda)| \leq \delta$ into a region containing $|\lambda| \geq M_2$ for some large M_2 and $|\arg(\lambda) - \frac{\pi}{2}| \leq \varepsilon_1$ for some $\varepsilon_1 > 0$. Also, $h_{m,\ell}(\cdot)$ is analytic in this region due to the analyticity of $f(0,a,\lambda)$ in Theorem 3.2. Following Sibuya, we will show that for every large positive integer n there exists λ_n such that

$$h_{m,\ell}(\lambda_n) = (2n+1)\pi i.\tag{7.3}$$

Next, Proposition 7.1 with $\mu = \lambda^{\frac{m+2}{2m}}$ implies that there exist $M_1' > 0$ and $0 < \delta' < \frac{\pi}{6}$ such that $\mu(1 + A(\mu))$ is univalent in the sector $|\arg(\mu)| \le \delta'$ and $|\mu| \ge M_1'$. Thus, for each $n \in \mathbb{Z}$, there is at most one μ_n in the sector $|\arg(\mu)| \le \delta'$ and $|\mu| \ge M_1'$ such that

$$\mu_n(1 + A(\mu_n)) = B_n, \tag{7.4}$$

where A is the error term in (7.2) and

$$B_n = \frac{(2n+1)\,\pi}{4K_m \cos\left(\frac{\pi}{m}\right)\sin\left(\frac{(2s-1)\pi}{m}\right)}.$$

Then there exists $M_1'' \ge M_1'$ such that in the sector $|\arg(\mu)| \le \delta'$ and $|\mu| \ge M_1''$,

$$|A(\mu)| < \frac{1}{5}.$$

Also we can take n large enough so that $\frac{1}{2}B_n \ge M_1''$. Next, we define

$$A_n := \{|A(\mu)| : |\arg(\mu)| \le \delta', |\mu| \ge B_n\}.$$

Then $\lim_{n\to\infty} A_n = 0$

Suppose that n is so large that $A_n \le \frac{1}{2}\sin(\delta') < \frac{1}{4}$. Then the disk defined by $|\mu - B_n| \le \frac{A_n}{1-2A_n}B_n$ is contained in the sector $|\arg(\mu)| \le \delta'$ and $|\mu| \ge \frac{1}{2}B_n$. Moreover, on the circle $|\mu - B_n| = \frac{A_n}{1-2A_n}B_n$,

$$\begin{aligned}
|\mu - B_n| - |\mu||A(\mu)| &\ge |\mu - B_n| - |\mu|A_n\\
&\ge |\mu - B_n|(1 - A_n) - A_nB_n\\
&\ge \frac{1-A_n}{1-2A_n}A_nB_n - A_nB_n > 0.
\end{aligned}$$

Thus, by the Rouché's theorem in complex analysis (see, e. g., [7, p. 125]), $\mu - B_n$ and $\mu(1 + A(\mu)) - B_n$ have the same number of zeros in the disk and hence, (7.4) has exactly one μ_n in the sector. Therefore, there exists $N = N(m, a) > 0$ such that (7.3) has exactly one solution λ_n for all integers $n \ge N$.

Next, since

$$\begin{aligned}
&-\frac{\nu(G^s(a))}{m}\ln(\omega^{2s}\lambda) + \frac{\nu(G^{-s}(a))}{m}\ln(\omega^{-2s}\lambda)\\
&-\frac{\nu(G^{s-1}(a))}{m}\ln(\omega^{2s-2}\lambda) + \frac{\nu(G^{-s+1}(a))}{m}\ln(\omega^{-2s+2}\lambda) - \frac{4\nu(G^s(a))}{m+2}\pi i\\
=&\frac{\nu(G^s(a))}{m}\left(-\ln(\omega^{2s}\lambda) + \ln(\omega^{-2s}\lambda) + \ln(\omega^{2s-2}\lambda) - \ln(\omega^{-2s+2}\lambda)\right) - \frac{4\nu(G^s(a))}{m+2}\pi i\\
=&-\frac{4}{m}\nu(G^s(a))\pi i,
\end{aligned}$$

using Corollary 3.3, (7.3) becomes

$$(2n+1)\,\pi i = \sum_{j=0}^{m+1} e_{\ell,j}(a)\lambda_n^{\frac{1}{2}+\frac{1-j}{m}} - \frac{4}{m}\nu(G^s(a))\pi i + O\left(\lambda^{-\rho}\right), \tag{7.5}$$

where

$$
\begin{aligned}
e_{\ell,j}(a) =& K_{m,j}(G^s(a))\omega^{2s\left(\frac{1}{2}+\frac{1-j}{m}\right)} - K_{m,j}(G^{-s}(a))\omega^{-2s\left(\frac{1}{2}+\frac{1-j}{m}\right)} \\
&+ K_{m,j}(G^{s-1}(a))\omega^{2(s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)} - K_{m,j}(G^{-s+1}(a))\omega^{-2(s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)} \\
=& K_{m,j}(a)\left(\omega^{-sj+2s\left(\frac{1}{2}+\frac{1-j}{m}\right)} - \omega^{sj-2s\left(\frac{1}{2}+\frac{1-j}{m}\right)}\right) \\
&+ K_{m,j}(a)\left(\omega^{-(s-1)j+2(s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)} - \omega^{(s-1)j-2(s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}\right) \\
=& 4iK_{m,j}(a)\sin\left(\frac{(1-j)\ell\pi}{m}\right)\cos\left(\frac{(1-j)\pi}{m}\right),
\end{aligned}
$$

where we used Lemma 3.5.

In summary, we have showed that for each $a \in \mathbb{C}^m$, there exists $N_0 = N_0(m,a) \leq N$ such that $\{\lambda_n\}_{n\geq N_0}$ is the set of all eigenvalues that satisfy (7.5), since there are only at most finitely many eigenvalues outside $|\arg(\lambda)| \leq \varepsilon$ or inside $|\lambda| \leq M$ for any $\varepsilon > 0$, $M > 0$. In order to complete proof, we need to show that N_0 does not depend on $a \in \mathbb{C}^m$.

We will prove that for each $R > 0$, if $|a| \leq R$, then $N_0(m,a) = N_0(m,0)$. The error terms in (3.5) and (3.6) are uniform on the closed ball $|a| \leq R$ and hence, so is $A(\mu)$ as $\mu \to \infty$ in the sector $|\arg(\mu)| \leq \delta'$. Thus, we can choose N independent of $|a| \leq R$. Since $\mu(1 + A(\mu))$ is univalent in the sector $|\arg(\mu)| \leq \delta'$ and $|\mu| \geq M_1'' \geq M_1'$, by the implicit function theorem and (7.4) the function $a \mapsto \mu_n(a)$ is continuous on $|a| \leq R$ for each $n \geq N$ and hence, so is $a \mapsto \lambda_n(a)$ for $n \geq N$.

Next, we claim that for all $|a| \leq R$, there are exactly the same number of eigenvalues that are not in $\{\lambda_n(a)\}_{n\geq N}$. To prove this, we will use the Hurwitz's theorem (see, e. g., [7, p. 152]) in complex analysis. That is, if a sequence of analytic functions converges uniformly to an analytic function on any compact sets, then eventually functions in the sequence and the limit function have the same number of zeros in any open set whose boundary does not contain any zeros of the limit function. The Hurwitz's theorem implies that since the eigenvalues are the zeros of the entire function $\mathcal{W}_{-s,s}(a,\lambda)$, they vary continuously as a and hence, there is no sudden appearance or disappearance of eigenvalues. Also, none of the eigenvalues that are not in $\{\lambda_n(a)\}_{n\geq N}$ can be continuously mapped to $\lambda_n(a)$ for some $n \geq N$ as a varies and hence, the claim is proved. This completes the proof. □

Next we prove Theorem 1.1 for $1 \leq \ell < \lfloor \frac{m}{2} \rfloor$ is even.

Proof of Theorem 1.1 when $1 \leq \ell < \lfloor \frac{m}{2} \rfloor$ is even. Proof is very similar to the case when $1 \leq \ell < \lfloor \frac{m}{2} \rfloor$ is odd. Let $\ell = 2s$ for some $s \in \mathbb{N}$.

Recall that λ is an eigenvalue of H_ℓ if and only if $\mathcal{W}_{-1,2s}(G^{-s+1}(\widetilde{a}), \omega^{-2s+1}\lambda) = 0$. Then from (5.3), we have

$$\begin{aligned}
&\exp\left[L(G^{s+1}(\widetilde{a}), \omega^{2s+1}\lambda) - L(G^{-s}(\widetilde{a}), \omega^{-2s-1}\lambda)\right]\\
&\times \exp\left[L(G^{s}(\widetilde{a}), \omega^{2s-1}\lambda) - L(G^{-s+1}(\widetilde{a}), \omega^{-2s+1}\lambda) + O\left(\lambda^{-\rho}\right)\right] = -1,
\end{aligned} \tag{7.6}$$

where $\widetilde{a} = G^{-\frac{1}{2}}(a)$ and where we used $[1 + O(\lambda^{-\rho})] = \exp[O(\lambda^{-\rho})]$ again.

Like in the case when $1 \leq \ell < \lfloor \frac{m}{2} \rfloor$ is odd, from Lemma 3.3, we have

$$(2n+1)\pi i = \sum_{j=0}^{m+1} e_{\ell,j}(a)\lambda_n^{\frac{1}{2}+\frac{1-j}{m}} + O\left(\lambda^{-\rho}\right), \tag{7.7}$$

where for $0 \leq j \leq m+1$,

$$\begin{aligned}
e_{\ell,j}(a) =& K_{m,j}(G^{s+\frac{1}{2}}(a))\omega^{(2s+1)\left(\frac{1}{2}+\frac{1-j}{m}\right)} - K_{m,j}(G^{-s-\frac{1}{2}}(a))\omega^{-(2s+1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}\\
&+ K_{m,j}(G^{s-\frac{1}{2}}(a))\omega^{(2s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)} - K_{m,j}(G^{-s+\frac{1}{2}}(a))\omega^{-(2s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}\\
=& \sum_{k=0}^{j}(-1)^k K_{m,j,k} b_{j,k}(a)\left(\omega^{-j\frac{\ell+1}{2}+(\ell+1)\left(\frac{1}{2}+\frac{1-j}{m}\right)} - \omega^{j\frac{\ell+1}{2}-(\ell+1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}\right.\\
&\qquad\left. + \omega^{-j\frac{\ell-1}{2}+(\ell-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)} - \omega^{j\frac{\ell-1}{2}-(\ell-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}\right)\\
=& 4i\sum_{k=0}^{j}(-1)^k K_{m,j,k} b_{j,k}(a)\sin\left(\frac{(1-j)\ell\pi}{m}\right)\cos\left(\frac{(1-j)\pi}{m}\right),
\end{aligned}$$

where we used Lemma 3.5 and $\ell = 2s$. Then we use the arguments in the case when ℓ is odd to show that there exists $N_0 = N_0(m)$ such that the set $\{\lambda_n\}_{n\geq N_0}$ of all eigenvalues satisfy (7.7) and hence, the proof is completed. □

8. Proof of Theorem 1.1 when $\ell = \lfloor \frac{m}{2} \rfloor$ and when $\frac{m}{2} < \ell \leq m-1$.

In this section, we prove Theorem 1.1 for $\ell = \lfloor \frac{m}{2} \rfloor$. We first prove the theorem when m is even, and later, we will treat the cases when m is odd. Then at the end of the section, we will prove the theorem when $\frac{m}{2} < \ell \leq m-1$, by scaling. Proof of existence of $N_0 = N_0(m)$ is the same as in Section 7, so below we will omit this part of proof.

8.1. When m is even.

We further divide the case into when ℓ is odd and when ℓ is even.

Proof of Theorem 1.1 when m is even and $\ell = \lfloor \frac{m}{2} \rfloor$ is odd. Suppose that m is even and $\ell = \frac{m}{2} = 2s-1$ for some $s \in \mathbb{N}$. Recall that λ is an eigenvalue of H_ℓ if and only if $\mathcal{W}_{-s,s}(a,\lambda) = 0$ if and only if $\mathcal{W}_{-1,2s-1}(G^{-s+1}(a), \omega^{-2s+2}\lambda) = 0$.

So if $\mathcal{W}_{-1,2s-1}(G^{-s+1}(a), \omega^{-2s+2}\lambda) = 0$ and if $|\lambda|$ is large enough, then from (5.14),

$$\begin{aligned}
&\exp\left[L(G^{3s-1}(a), \omega^{2s-2}\lambda) - L(G^{s+1}(a), \omega^{-2s+2}\lambda)\right]\\
&\times \exp\left[L(G^{s-1}(a), \omega^{2s-2}\lambda) - L(G^{-s+1}(a), \omega^{-2s+2}\lambda) + O\left(\lambda^{-\rho}\right)\right] = -\omega^{4\nu(G^s(a))}.
\end{aligned}$$

Then by Lemma 3.3,

$$\left(\frac{8\nu(G^s(a))}{m+2}+2n+1\right)\pi i=\sum_{j=0}^{m+1}e_{\ell,j}(a)\lambda_n^{\frac{1}{2}+\frac{1-j}{m}}+\frac{16(s-1)\nu(G^s(a))}{m(m+2)}\pi i+O\left(\lambda^{-\rho}\right),$$

where for $0\le j\le m+1$, the coefficients $e_{\ell,j}(a)$ are given by

$$\begin{aligned}e_{\ell,j}(a)=&K_{m,j}(G^{3s-1}(a))\omega^{2(s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}-K_{m,j}(G^{s+1}(a))\omega^{-2(s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}\\&+K_{m,j}(G^{s-1}(a))\omega^{2(s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}-K_{m,j}(G^{-s+1}(a))\omega^{-2(s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}\\=&K_{m,j}(a)\Big(\omega^{(s+1)j+2(s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}-\omega^{-(s+1)j-2(s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}\\&\qquad+\omega^{-(s-1)j+2(s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}-\omega^{(s-1)j-2(s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}\Big)\\=&2iK_{m,j}(a)\left(1+(-1)^j\right)\sin\left(\frac{(1-j)(2s-2)\pi}{m}\right)\\=&4iK_{m,j}(a)\sin\left(\frac{(1-j)\pi}{2}\right)\cos\left(\frac{(1-j)\pi}{m}\right),\end{aligned}$$

where we used Lemma 3.5. □

Proof of Theorem 1.1 when m is even and $\ell=\lfloor\frac{m}{2}\rfloor$ is even. Let $\ell=\frac{m}{2}=2s$ for some $s\in\mathbb{N}$. Recall that λ is an eigenvalue of H_ℓ if and only if $\mathcal{W}_{-1,2s}(G^{-s+1}(\widetilde{a}),\omega^{-2s+1}\lambda)=0$.

Suppose that $\mathcal{W}_{-1,2s}(G^{-s+1}(\widetilde{a}),\omega^{-2s+1}\lambda)=\mathcal{W}_{-1,2s}(G^{-s+\frac{1}{2}}(a),\omega^{-2s+1}\lambda)=0$. Then by (5.14),

$$\begin{aligned}&\exp\left[L(G^{3s+\frac{1}{2}}(a),\omega^{2s-1}\lambda)-L((G^{s+\frac{3}{2}}(a),\omega^{-2s+1}\lambda)\right]\\&\times\exp\left[L(G^{s-\frac{1}{2}}(a),\omega^{2s-1}\lambda)-L(G^{-s+\frac{1}{2}}(a),\omega^{-2s+1}\lambda)+O\left(\lambda^{-\rho}\right)\right]=-1.\end{aligned}$$

Then like before,

$$\left(2n+1\right)\pi i=\sum_{j=0}^{m+1}e_{\ell,j}(a)\lambda_n^{\frac{1}{2}+\frac{1-j}{m}}+O\left(\lambda^{-\rho}\right),$$

where for $0\le j\le m+1$,

$$\begin{aligned}e_{\ell,j}(a)=&K_{m,j}(G^{3s+\frac{1}{2}}(a))\omega^{(2s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}-K_{m,j}((G^{s+\frac{3}{2}}(a))\omega^{-(2s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}\\&+K_{m,j}(G^{s-\frac{1}{2}}(a))\omega^{(2s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}-K_{m,j}(G^{-s+\frac{1}{2}}(a))\omega^{-(2s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}\\=&2i\sum_{k=0}^{j}(-1)^kK_{m,j,k}b_{j,k}(a)\left(1+(-1)^j\right)\sin\left(\frac{(1-j)(2s-1)\pi}{m}\right)\\=&4i\sum_{k=0}^{j}(-1)^kK_{m,j,k}b_{j,k}(a)\sin\left(\frac{(1-j)\pi}{2}\right)\cos\left(\frac{(1-j)\pi}{m}\right).\end{aligned}$$

□

8.2. **When m is odd.** We divide the case into when ℓ is odd and when ℓ is even.

Proof of Theorem 1.1 when m and $\ell = \lfloor \frac{m}{2} \rfloor$ are odd. Let m and $\ell = \frac{m-1}{2} = 2s-1$ be odd. Suppose that λ is an eigenvalue of H_ℓ. Since λ is an eigenvalue of H_ℓ if and only if $\mathcal{W}_{-1,2s-1}(G^{-s+1}(a), \omega^{-2s+2}\lambda) = 0$, if $|\lambda|$ is large enough, then from (5.16),

$$\begin{aligned}
&\exp\left[L(G^{s-1}(a), \omega^{2s-2}\lambda) - L(G^{-s+1}(a), \omega^{-2s+2}\lambda)\right] \\
&\times \exp\left[-L(G^{3s+1}(a), \omega^{-2s}\lambda) - L(G^{s+1}(a), \omega^{-2s+1}\lambda) + O\left(\lambda^{-\rho}\right)\right] = -1.
\end{aligned}$$

Then, from Lemma 3.3,

$$(2n+1)\,\pi i = \sum_{j=0}^{m+1} e_{\ell,j}(a)\lambda_n^{\frac{1}{2}+\frac{1-j}{m}} + O\left(\lambda^{-\rho}\right),$$

where for $0 \le j \le m+1$,

$$\begin{aligned}
e_{\ell,j}(a) =& K_{m,j}(G^{s-1}(a))\omega^{(2s-2)\left(\frac{1}{2}+\frac{1-j}{m}\right)} - K_{m,j}(G^{-s+1}(a))\omega^{(-2s+2)\left(\frac{1}{2}+\frac{1-j}{m}\right)} \\
&- K_{m,j}(G^{3s+1}(a))\omega^{-2s\left(\frac{1}{2}+\frac{1-j}{m}\right)} - K_{m,j}(G^{s+1}(a))\omega^{(-2s+1)\left(\frac{1}{2}+\frac{1-j}{m}\right)} \\
=& K_{m,j}(a)\left(\omega^{-j(s-1)+(2s-2)\left(\frac{1}{2}+\frac{1-j}{m}\right)} - \omega^{j(s-1)-(2s-2)\left(\frac{1}{2}+\frac{1-j}{m}\right)}\right. \\
&\qquad \left. -\omega^{-j(3s+1)-2s\left(\frac{1}{2}+\frac{1-j}{m}\right)} - \omega^{-j(s+1)-(2s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)}\right) \\
=& K_{m,j}(a)\left(\omega^{(1-j)(2s-2)\left(\frac{1}{2}+\frac{1}{m}\right)} - \omega^{-(1-j)(2s-2)\left(\frac{1}{2}+\frac{1}{m}\right)}\right. \\
&\qquad \left. -\omega^{js-2s\left(\frac{1}{2}+\frac{1-j}{m}\right)} + \omega^{-js+2s\left(\frac{1}{2}+\frac{1-j}{m}\right)}\right) \\
=& 2iK_{m,j}(a)\left(\sin\left(\frac{(m-3)(1-j)\pi}{2m}\right) + \sin\left(\frac{(m+1)(1-j)\pi}{2m}\right)\right) \\
=& 4iK_{m,j}(a)\sin\left(\frac{(1-j)\ell\pi}{m}\right)\cos\left(\frac{(1-j)\pi}{m}\right).
\end{aligned}$$

□

Proof of Theorem 1.1 when m is odd and $\ell = \lfloor \frac{m}{2} \rfloor$ is even. Let $\ell = \frac{m-1}{2} = 2s$ for some $s \in \mathbb{N}$. Then λ is an eigenvalue of H_ℓ if and only if $\mathcal{W}_{-1,2s}(G^{-s+1}(\widetilde{a}), \omega^{-2s+1}\lambda) = 0$.

If $\mathcal{W}_{-1,2s}(G^{-s+1}(\widetilde{a}), \omega^{-2s+1}\lambda) = 0$ and if $|\lambda|$ is large enough, then from (5.16),

$$\begin{aligned}
&\exp\left[L(G^{s}(\widetilde{a}), \omega^{2s-1}\lambda) - L(G^{-s+1}(\widetilde{a}), \omega^{-2s+1}\lambda)\right] \\
&\times \exp\left[-L(G^{3s+3}(\widetilde{a}), \omega^{-2s-1}\lambda) - L(G^{s+2}(\widetilde{a}), \omega^{-2s}\lambda) + O\left(\lambda^{-\rho}\right)\right] = -1.
\end{aligned}$$

Then, from Lemma 3.3,

$$(2n+1)\,\pi i = \sum_{j=0}^{m+1} e_{\ell,j}(a)\lambda_n^{\frac{1}{2}+\frac{1-j}{m}} + O\left(\lambda^{-\rho}\right),$$

where for $0 \leq j \leq m+1$,

$$\begin{aligned} e_{\ell,j}(a) =& K_{m,j}(G^{s-\frac{1}{2}}(a))\omega^{(2s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)} - K_{m,j}((G^{-s+\frac{1}{2}}(a))\omega^{-(2s-1)\left(\frac{1}{2}+\frac{1-j}{m}\right)} \\ & - K_{m,j}(G^{3s+\frac{5}{2}}(a))\omega^{-(2s+1)\left(\frac{1}{2}+\frac{1-j}{m}\right)} - K_{m,j}(G^{s+\frac{3}{2}}(a))\omega^{-2s\left(\frac{1}{2}+\frac{1-j}{m}\right)} \\ =& 2i\sum_{k=0}^{j}(-1)^k K_{m,j,k} b_{j,k}(a)\left[\sin\left(\frac{(1-j)(2s-1)\pi}{m}\right) + \sin\left(\frac{(1-j)(2s+1)\pi}{m}\right)\right] \\ =& 4i\sum_{k=0}^{j}(-1)^k K_{m,j,k} b_{j,k}(a)\sin\left(\frac{(1-j)\ell\pi}{m}\right)\cos\left(\frac{(1-j)\pi}{m}\right). \end{aligned}$$

□

Theorem 1.1 for $1 \leq \ell \leq \frac{m}{2}$ has been proved. Next we prove Theorem 1.1 for $\frac{m}{2} < \ell \leq m-1$, by the change of the variables $z \mapsto -z$.

8.3. When $\frac{m}{2} < \ell \leq m-1$.

Proof of Theorem 1.1 when $\frac{m}{2} < \ell \leq m-1$. Suppose that $\ell > \frac{m}{2}$. If u is an eigenfunction of H_ℓ, then $v(z) = u(-z, \lambda)$ solves

$$-v''(z) + \left[(-1)^{-\ell}(-iz)^m - P(-iz)\right] v(z) = \lambda v(z),$$

and

$$v(z) \to 0 \text{ exponentially, as } z \to \infty \text{ along the two rays} \quad \arg z = -\frac{\pi}{2} \pm \frac{((m-\ell)+1)\pi}{m+2}.$$

The coefficient vector of $P(-z)$ is $((-1)^{m-1}a_1, (-1)^{m-2}a_2, \dots, (-1)^1 a_{m-1}, a_m)$. Certainly,

$$\sin\left(\frac{(1-j)(m-\ell)\pi}{m}\right) = (-1)^j \sin\left(\frac{(1-j)\ell\pi}{m}\right).$$

Also, one can find from Lemma 3.5 that for $0 \leq k \leq j$,

$$b_{j,k}((-1)^{m-1}a_1, (-1)^{m-2}a_2, \dots, -a_{m-1}, a_m) = (-1)^{mk-j} b_{j,k}(a_1, a_2, \dots, a_{m-1}, a_m).$$

Moreover, $c_{m-\ell,j}((-1)^{m-1}a_1, (-1)^{m-2}a_2, \dots, -a_{m-1}, a_m) = c_{\ell,j}(a_1, a_2, \dots, a_{m-1}, a_m)$. This completes proof of Theorem 1.1. □

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